International Mathematics

TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Spring 2015**

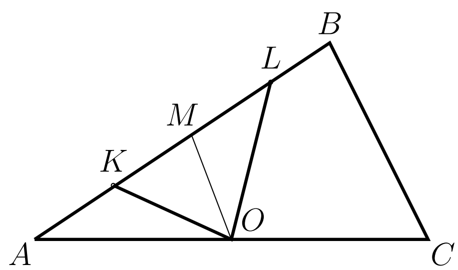
Problem 1. Is it possible to paint six faces of a cube into three colours so that each colour is present, but from any position one can see at most two colours?

Answer. Yes, it is possible.

Solution. Colour two opposite faces of the cube in red and blue, while the other faces in green. From any position one can not see red and blue faces at the same time. □

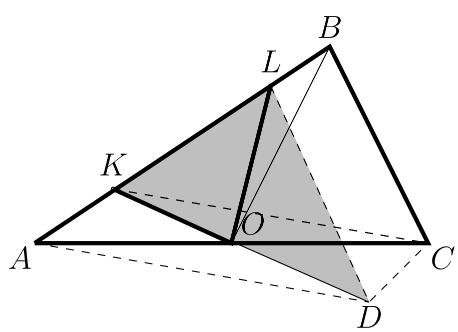
Problem 2. Points K and L are marked on side AB of triangle ABC so that $KL = BC$ and $AK = LB$. Given that O is the midpoint of side AC, prove that $\angle KOL = 90°$.

Solution 1. Let M be a midpoint of AB. Then $MO = 1/2BC = 1/2KL = KM = ML$. Therefore, points K, M, and O belong to a circle with radius KM and centre at M. Since KL is a diameter of this circle, $\angle KOL = 90°$.



□

Solution 2. Let us draw $LD \parallel BC$ and $CD \parallel AB$. Quadrilateral $AKCD$ is a parallelogram ($CD = LB = AK$ and $CD \parallel AK$). Then O, the midpoint of AC is the point of intersection of its diagonals and therefore $KO = OD$. Since the triangle KLD is isosceles ($LK = BC = LD$, its median LO is also an altitude. Hence $\angle KOL = 90°$.



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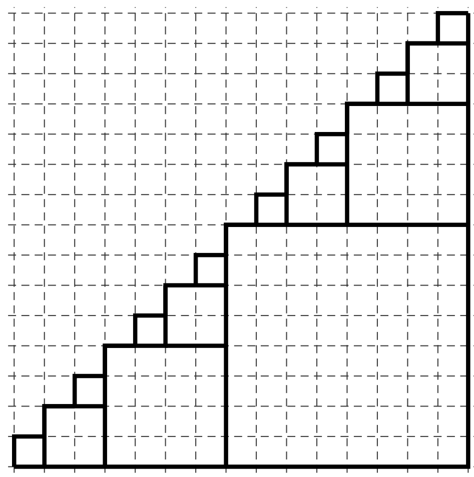
Problem 3. Pete summed up 10 consecutive powers of 2, starting from some power, while Basil summed up several consecutive positive integers starting from 1. Can they get the same result?

Answer. Yes, they can.

Solution. Indeed, let $2^{k+1} + \ldots 2^{k+10} = 1 + 2 + \cdots + n$. Simplifying we see that $2^{k+2}(2^{10} - 1) = n(n+1)$ holds for $n = 2^{10} - 1$ and $k = 8$. □

Problem 4. A figure, given on the grid, consists of a 15-step staircase and horizontal and vertical bases (see the figure). What is the least number of squares one can split this figure into? (Splitting is allowed only along the grid).

Solution. Note that each step's corner belongs to some square and no two corners belong to the same square. Therefore the number of squares is no less than 15. Example that splitting the figure into 15 squares can be achieved:



□

Problem 5. Among $2n + 1$ positive integers there is exactly one 0, while each of the numbers $1, 2, ..., n$ is presented exactly twice. For which n can one line up these numbers so that for any $m = 1, \ldots, n$ there are exactly m numbers between two m's?

Answer. For any n.

Solution. Observe that two sets of odd numbers, each set from from 1 to $2k + 1$ can be arranged according to the requirement with one empty space in the middle:

$$2k + 1, 2k - 1, \ldots, 3, 1, \square, 1, 3, \ldots, 2k - 1, 2k + 1$$

while two sets of even from from 1 to $2k$ can be arranged according to the requirement with two empty spaces in the middle:

$$2k, 2k - 2, \ldots, 2, 1, \square\square, 1, 2, \ldots, 2k - 2, 2k$$

(a) $n = 2k + 1$. Consider the follow arrangement:

$$2k{+}1, 2k{-}1, \dots, 3, 1, \boxed{2k}, 1, 3, \dots, 2k{-}1, 2k{+}1, 2k{-}2, 2k{-}4 \dots 2, \boxed{2k, 0}, 2, \dots, 2k{-}2$$

Inserting two copies of $2k$ as shown, we see that for any $m \neq 2k$ requirement holds and we can check that it holds for $m = 2k$ as well.

Indeed,

$$1, 3, \dots, 2k - 1, 2k + 1, 2k - 2, 2k - 4 \dots 2,$$

includes $k + 1$ of odd numbers and $k - 1$ of even numbers, $2k$ numbers in total.

(b) $n = 2k$. In a similar way one can check that the following arrangement works:

$$2k - 1, 2k - 3, \dots, 3, 1, \boxed{2k}, 1, 3, \dots, 2k - 1, 2k - 2, 2k - 4 \dots 2, \boxed{0, 2k}, 2, \dots, 2k - 2$$

. $\square$