

**International Mathematics**  
**TOURNAMENT OF THE TOWNS**

**Junior A-Level Paper**

**Spring 2015**

- 1 [4] A point is chosen inside a parallelogram  $ABCD$  so that  $CD = CE$ . Prove that the segment  $DE$  is perpendicular to the segment connecting the midpoints of the segments  $AE$  and  $BC$ .
- 2 [6] Area 51 has the shape of a non-convex polygon. It is protected by a chain fence along its perimeter and is surrounded by a minefield so that a spy can only move along the fence. The spy went around the Area once so that the Area was always on his right. A straight power line with 36 poles crosses this area so that some of the poles are inside the Area, and some are outside it. Each time the spy crossed the power line, he counted the poles to the left of him (he could see all the poles). Having passed along the whole fence, the spy had counted 2015 poles in total. Find the number of poles inside the fence.
- 3 (a) [3] The integers  $x$ ,  $x^2$  and  $x^3$  begin with the same digit. Does it imply that this digit is 1?  
(b) [4] The same question for the integers  $x$ ,  $x^2$ ,  $x^3$ ,  $\dots$ ,  $x^{2015}$ .
- 4 For each side of some polygon, the line containing it contains at least one more vertex of this polygon. Is it possible that the number of vertices of this polygon is  
(a) [4]  $\leq 9$ ?  
(b) [5]  $\leq 8$ ?
- 5 (a) [4] A  $2 \times n$ -table (with  $n > 2$ ) is filled with numbers so that the sums in all the columns are different. Prove that it is possible to permute the numbers in the table so that the sums in the columns would still be different and the sums in the rows would also be different.  
(b) [5] A  $10 \times 10$ -table is filled with numbers such that the sums in all the columns are different. Is it always possible to permute the numbers in the table so that the sums in the columns would still be different and the sums in the rows would also be different?
- 6 [9] A convex  $N$ -gon with equal sides is located inside a circle. Each side is extended in both directions up to the intersection with the circle so that it contains two new segments outside the polygon. Prove that one can paint some of these new  $2N$  segments in red and the rest in blue so that the sum of lengths of all the red segments would be the same as for the blue ones.
- 7 [10] An Emperor invited 2015 wizards to a festival. Each of the wizards knows who of them is good and who is evil, however the Emperor doesn't know this. A good wizard always tells the truth, while an evil wizard can tell the truth or lie at any moment. The Emperor asks each wizard (in an order of his choice) a single question, maybe different for different wizards, and listens to the answer which is either "yes" or "no". Having listened to all the answers, the Emperor expels a single wizard through a magic door which shows if this wizard is good or evil. Then the Emperor repeats the procedure with the remaining wizards, and so on. The Emperor may stop at any moment, and after this the Emperor may expel or not expel a wizard. Prove that the Emperor can expel all the evil wizards having expelled at most one good wizard.