37th International Mathematics

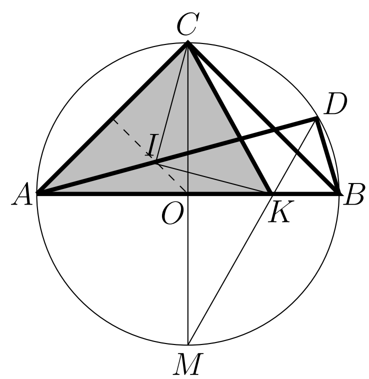
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2015**

Problem 1. Let p be a prime number. Determine the number of positive integers n such that pn is a multiple of $p+n$.

Solution. Let k be a positive integer such that $pn = k(n+p)$. Then $pn = kn+kp$ and $n-k = kn/p$. Note that $k < n$ and $k < p$ ($n-k$ must be positive while kn/p can not exceed n). It follows that $n = mp$, where m is a positive integer. Thus, $pm-k = km$ so that $p = k(m+1)/m$ and therefore $k = lm$, where l is a positive integer. Then $p = l(m+1)$ which implies that $l = 1$ (p is prime). Hence $m = p-1$ and $n = p(p-1)$ is the only one possible value. □

Problem 2. Suppose that ABC and ABD are right-angled triangles with common hypotenuse AB (D and C are on the same side of line AB). If $AC = BC$ and DK is a bisector of angle ADB, prove that the circumcenter of triangle ACK belongs to line AD.



Solution. Observe that triangles ABC and ADB share the same circumcircle with diameter AB. Denote by M a point symmetrical to C about the centre. Observe that K belongs to MD. (Indeed $AM = BM$ implies that $\angle ADM = \angle MDB$).

Let $\angle CAD = \alpha$. Then $\angle CMD = \angle CAD = \alpha$. Since triangle CMK is isosceles its altitude OK is also a bisector and therefore $\angle AKC = 90^\circ - \alpha$. Let I be a point on AD equidistant from A and C. Since $\angle ACI = \angle CAI = \alpha$, $\angle AIC = 180^\circ - 2\alpha$. Consider a circle with centre I and radius $AI = CI$. Since $\angle AKC = 1/2\angle AIC$, K belongs to this circle. Hence I is a centre of circumcircle of triangle ACK. □

Problem 3. Three players play the game "rock-paper-scissors". In every round, each player simultaneously shows one of these shapes. Rock beats scissors, scissors beat paper, while paper beats rock. If in a round exactly two distinct shapes are shown (and thus one of them is shown twice) then 1 point is added to the score of the player(s) who showed the winning shape, otherwise no point is added. After several rounds it occurred that each shape had been shown the same number of times. Prove that the total sum of points at this moment was a multiple of 3.

Solution. Outcome in a round is a triple of shapes and depending on the number of distinct shapes can be one of three kinds: (x, x, x), (x, y, z), and (x, x, y), where x, y and z stand for shapes.

Note that outcomes (x, x, x), (x, y, z) worth no points so if by the final moment of the game no other outcomes appear, then the total number of points $P = 0$. Assume there is an outcome (x, x, y). Then there is at least one more outcome from this category (when exactly two shapes are the same). We will show the way of replacing two outcomes of this sort by two new triples, one of which worths no point while preserving the number of points by modulo 3 (In other words P modulo 3 is invariant under operation of rearrangement).

Assume that $x < y$. Then $y < z$ and $z < x$ so that we have $P(x, x, x) = 0$, $P(y, y, y) = 0$, $P(z, z, z) = 0$, $P(x, x, y) = 1$, $P(x, y, y) = 2$, $P(x, z, z) = 1$, $P(x, x, z) = 2$, $P(y, y, z) = z$, $P(y, z, z) = 2$.

Below the list all possible cases for the second triple.

Case (x, x, y). $(x, x, y) + (x, x, y) \to (x, x, x) + (x, y, y)$. $\Delta P \equiv 0 \mod 3$.

(ΔP is the difference between the numbers of the total points before and after rearrangement).

Case (x, y, y). $(x, x, y) + (x, y, y) \to (x, x, x) + (x, y, y)$. $\Delta P \equiv 0 \mod 3$.

Case (x, x, z). $(x, x, y) + (x, x, z) \to (x, x, x) + (x, y, z)$. $\Delta P \equiv 0 \mod 3$.

Case (x, z, z). $(x, x, y) + (x, z, z) \to (x, x, x) + (y, z, z)$. $\Delta P \equiv 0 \mod 3$.

Case (y, y, z). $(x, x, y) + (y, y, z) \to (y, y, y) + (x, x, z)$. $\Delta P \equiv 0 \mod 3$.

Case (y, z, z). $(x, x, y) + (y, z, z) \to (x, y, z) + (x, y, z)$. $\Delta P \equiv 0 \mod 3$.

Since the number of triples of the third category keeps decreasing eventually we come to a situation when no such elements left. Since P is invariant and we started with $P \equiv \mod 3$, then at the final moment $P \equiv 0 \mod 3$. □

Solution 2. (Hessami Elnaz). Let r, s, p represent rock, scissors and paper respectively. We can exclude from consideration outcomes (r, p, s) since they contribute evenly in a total of each shape and worth no points. Assume that combinations (r, r, p), (r, r, s),(r, r, s), (s, s, r), (s, s, p), (p, p, r), (p, p, s), (r, r, r), (p, p, p), (s, s, s) appeared $b, c, d, e, f, g, u, v, w$ times respectively. Then the total number of points $P = b + d + g + 2c + 2e + 2f$.

According to the condition in the end of the game all shapes appeared in equal numbers. Thus we have:

$$\begin{aligned} 2b + 2c + d + f + 3u &= n, \\ b + e + 2f + 2g + 3v &= n, \\ c + 2d + 2e + g + 3w &= n \end{aligned}$$

Adding the first and the second doubled equation we have

$$\begin{aligned} &4b + 2c + 2e + 5f + d + 4g + 3u + 6v = 3n \implies \\ &b + d + g + 2c + 2e + 2f \equiv 0 \mod 3. \end{aligned}$$

Hence, $P \equiv 0 \mod 3$. □

Problem 4. In a country there are 100 cities. Every two cities are connected by a direct flight (in both directions). Each flight costs a positive (not necessarily integer) number of doubloons. The flights in both directions between two given cities are of the same cost. The average cost of a flight is 1 doubloon. A traveller plans to visit any m cities for m flights, starting and ending at his native city (which must be one of these m cities). Can the traveller always fulfil his plans given that he can spend at most m doubloons if

(a) $m = 99$;

(b) $m = 100$?

Solution. Let $A_1, A_2, \ldots A_{100}$ denote cities and let A_1 be a home town of the traveller.

(a) $m = 99$. The traveller can not fulfil his plans for sure. Example.

Let the cost of each flight connecting A_1 with every other city be $p = \$43$ while the cost of each of the remaining flights (connecting A_i and A_j, $i \neq 1, j \neq 1$) be $q = \$1/7$. Then the average of one flight is $2(99p + 99 \times 98q)/100 \times 99 = 1$ while the cost of any route including m cities which starts and ends at A_1 is $(2p + 97q)/99 > 99$.

(b) $m = 100$. *(Steven Chow)* In this case a route is a loop consisting of 100 cities. The routes differ only by the order of cities. No route passes through the same city twice. The traveller's home city is no special so he can start his route at any city.

Consider the total number of doubloons that the traveller must spent for every possible route. An average cost of one route is the sum of these costs divided by the number of possible routes.

Since every flight comes the same number of times in the total number of possible routes (it follows from symmetry of the situation), and since the average cost of one flight is 1 doubloon, the average cost of one route is 100 doubloons.

Then there is a route with the cost not exceeding 100 doubloons. Hence the traveller can always fulfil his plans.

□

Problem 5. An infinite increasing arithmetical progression is given. A new sequence is constructed in the following way: its first term is the sum of several first terms of the original sequence, its second term is the sum of several next terms of the original sequence and so on. Is it possible that the new sequence is a geometrical progression?

Solution. Example.

Arithmetical sequence: $1 + 2 + 3 + \cdots + n + \ldots$.

Geometrical sequence: $1 + 9 + 9^2 + \cdots + 9^k + \ldots$.

Let us show that for any $k = 0, 1, \ldots,$ there exists such n that

$$1 + 2 + 3 + \cdots + n = 1 + 9 + 9^2 + \cdots + 9^k.$$

which is equivalent to $(9^{k+1} - 1)/8 = n(n+1)/2$. Solving this equation we find that $n = (3^{k+1} - 1)/2$. □