37th International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Fall 2015**

1 [**3**] A geometrical progression consists of 37 positive integers. The first and the last terms are relatively prime numbers. Prove that the 19th term of the progression is the 18th power of some positive integer.

2 [**6**] A 10×10 square on a grid is split by 80 unit grid segments into 20 polygons of equal area (no one of these segments belongs to the boundary of the square). Prove that all polygons are congruent.

3 [**6**] Each coefficient of a polynomial is an integer with absolute value not exceeding 2015. Prove that every positive root of this polynomial exceeds 1/2016.

4 [**7**] Let $ABCD$ be a cyclic quadrilateral, K and N be the midpoints of the diagonals and P and Q be points of intersection of the extensions of the opposite sides. Prove that $\angle PKQ + \angle PNQ = 180°$.

5 Several distinct real numbers are written on a blackboard. Peter wants to create an algebraic expression such that among its values there would be these and only these numbers. He may use any real numbers, brackets, signs $+$, $-$, $\times$ and a special sign $\pm$. Usage of $\pm$ is equivalent to usage of $+$ and $-$ in all possible combinations. For instance, the expression 5 ± 1 results in $\{4, 6\}$, while $(2 \pm 0.5) \pm 0.5$ results in $\{1, 2, 3\}$. Can Peter construct an expression if the numbers on the blackboard are:

- **(a)** [**2**] 1, 2, 4?
- **(b)** [**6**] any 100 distinct real numbers?

6 Basil has a melon in a shape of a ball, 20 cm in diameter. Using a long knife, Basil makes three mutually perpendicular cuts. Each cut carves a circular segment in a plane of the cut, h cm deep (h is a height of the segment). Does it necessarily follow that the melon breaks into two or more pieces if

- **(a)** [**6**] $h = 17$?
- **(b)** [**6**] $h = 18$?

7 [**12**] N children no two of the same height stand in a line. The following two-step procedure is applied: first, the line is split into the least possible number of groups so that in each group all children are arranged from the left to the right in ascending order of their heights (a group may consist of a single child). Second, the order of children in each group is reversed, so now in each group the children stand in descending order of their heights. Prove that in result of applying this procedure $(N - 1)$ times the children in the line would stand from the left to the right in descending order of their heights.