37th International Mathematics

TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2015**

Problem 1. Is it true that every positive integer can be multiplied by one of the digits 1, 2, 3, 4 or 5 so that the resulting number starts with 1?

Solution. Let the number a start with digit x. We are going to check that for any a by appropriate choice of n from $\{1, 2, 3, 4, 5\}$ the product of $a \times n$ starts with 1.

(i) $x = 1$. We chose $n = 1$.

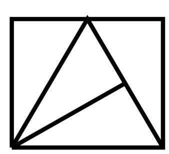
(ii) $x = 2, 3$. We choose $n = 5$. Indeed, the smallest value of the product $a \times 5$ is $20 \dots 0 \times 5$ while the largest value of the product does not exceed $39 \dots 9 \times 5 = (40 \dots 0 - 1) \times 5$. Both of these numbers start with digit 1, so do all the numbers in between.

(iii) $x = 4$. We can choose $n = 3$. The smallest value of $a \times n$ is no less than $40 \dots 0 \times 3$ while the largest number does not exceed $4 \dots 9 = (50 \dots 0 - 1) \times 3$. Both of these numbers as well as all numbers in between start with digit 1.

(iv) $x = 5, 6, 7, 8, 9$. We choose $n = 2$. The smallest value of $a \times n$ is no less than $50 \dots 0 \times 2$ while the largest number does not exceed $9 \dots 9 = (10 \dots 0 - 1) \times 2$. Both of these numbers as well all numbers in between start with digit 1. □

Problem 2. A rectangle is split into equal non-isosceles right-angled triangles (without gaps or overlaps). Is it true that any such arrangement contains a rectangle made of two such triangles?

Solution. Counterexample. Rectangle with sides $2, \sqrt{3}$ can be split into four non-isosceles right angle triangles with angles 30° and 60° as shown on the picture. No two triangles are arranged into rectangle.



□

Problem 3. Three players play the game "rock-paper-scissors". In every round, each player simultaneously shows one of these shapes. Rock beats scissors, scissors beat paper, while paper beats rock. If in a round exactly two distinct shapes are shown (and thus one of them is shown twice) then 1 point is added to the score of the player(s) who showed the winning shape, otherwise no point is added. After several rounds it occurred that each shape had been shown the same number of times. Prove that the total sum of points at this moment was a multiple of 3.

Solution. Outcome in a round is a triple of shapes and depending on the number of distinct shapes can be one of three kinds: (x,x,x), (x,y,z), and (x,x,y), where x, y and z stand for shapes.

Note that outcomes (x,x,x), (x,y,z) worth no points so if by the final moment of the game no other outcomes appear, then the total number of points $P = 0$. Assume there is an outcome (x,x,y). Then there is at least one more outcome from this category (when exactly two shapes are the same). We will show the way of replacing two outcomes of this sort by two new triples, one of which worths no point while preserving the number of points by modulo 3 (In other words P modulo 3 is invariant under operation of rearrangement).

Assume that $x < y$. Then $y < z$ and $z < x$ so that we have $P(x,x,x) = 0$, $P(y,y,y) = 0$, $P(z,z,z) = 0$, $P(x,x,y) = 1$, $P(x,y,y) = 2$, $P(x,z,z) = 1$, $P(x,x,z) = 2$, $P(y,y,z) = z$, $P(y,z,z) = 2$.

Below the list all possible cases for the second triple.

Case (x,x,y). $(x,x,y) + (x,x,y) \to (x,x,x) + (x,y,y)$. $\Delta P \equiv 0 \mod 3$.
(ΔP is the difference between the numbers of the total points before and after rearrangement).

Case (x,y,y). $(x,x,y) + (x,y,y) \to (x,x,x) + (x,y,y)$. $\Delta P \equiv 0 \mod 3$.

Case (x,x,z). $(x,x,y) + (x,x,z) \to (x,x,x) + (x,y,z)$. $\Delta P \equiv 0 \mod 3$.

Case (x,z,z). $(x,x,y) + (x,z,z) \to (x,x,x) + (y,z,z)$. $\Delta P \equiv 0 \mod 3$.

Case (y,y,z). $(x,x,y) + (y,y,z) \to (y,y,y) + (x,x,z)$. $\Delta P \equiv 0 \mod 3$.

Case (y,z,z). $(x,x,y) + (y,z,z) \to (x,y,z) + (x,y,z)$. $\Delta P \equiv 0 \mod 3$.

Since the number of triples of the third category keeps decreasing eventually we come to a situation when no such elements left. Since P is invariant and we started with $P \equiv \mod 3$, then at the final moment $P \equiv 0 \mod 3$. □

Solution 2. (Hessami Elnaz). Let r, s, p represent rock, scissors and paper respectively. We can exclude from consideration outcomes (r,p,s) since they contribute evenly in a total of each shape and worth no points. Assume that combinations (r,r,p), (r,r,s),(r,r,s), (s,s,r), (s,s,p), (p,p,r), (p,p,s), (r,r,r), (p,p,p), (s,s,s) appeared $b, c, d, e, f, g, u, v, w$ times respectively. Then the total number of points $P = b + d + g + 2c + 2e + 2f$.

According to the condition in the end of the game all shapes appeared in equal numbers. Thus we have:

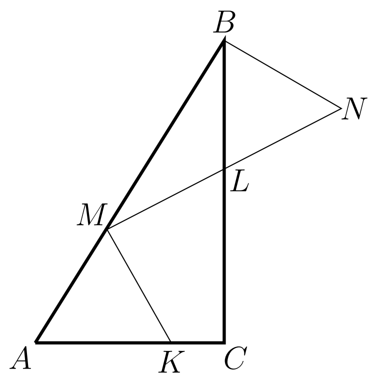
$$2b + 2c + d + f + 3u = n,$$
$$b + e + 2f + 2g + 3v = n,$$
$$c + 2d + 2e + g + 3w = n$$

Adding the first and the second doubled equation we have

$$4b + 2c + 2e + 5f + d + 4g + 3u + 6v = 3n \implies$$
$$b + d + g + 2c + 2e + 2f \equiv 0 \mod 3.$$

Hence, $P \equiv 0 \mod 3$. □

Problem 4. In a right-angled triangle ABC ($\angle C = 90°$) points K, L and M are chosen on sides AC, BC and AB respectively so that $AK = BL = a$, $KM = LM = b$ and $\angle KML = 90°$. Prove that $a = b$.



Solution. On extension of ML above point L mark point N such that $LN = ML$. Since quadrilateral $KMLC$ is cyclic ($\angle KML + \angle LCK = 180°$), $\angle BLN = \angle AKM$. Therefore triangles AMK and LBN are congruent (S-A-S). Then $\angle LBN = \angle MAK$. It follows that $\angle MBN = 90°$. Then L is a centre of the circumcircle of triangle MBN. Hence, $BL = ML$ as radiuses of the same circle. □

Problem 5. In a country there are 100 cities. Every two cities are connected by a direct flight (in both directions). Each flight costs a positive (not necessarily integer) number of doubloons. The flights in both directions between two given cities are of the same cost. The average cost of a flight is 1 doubloon. A traveller plans to visit any m cities for m flights, starting and ending at his native city (which must be one of these m cities). Can the traveller always fulfil his plans given that he can spend at most m doubloons if

(a) $m = 99$;

(b) $m = 100$?

Solution. Let $A_1, A_2, \ldots A_{100}$ denote cities and let A_1 be a home town of the traveller.

(a) $m = 99$. The traveller can not fulfil his plans for sure. Example.

Let the cost of each flight connecting A_1 with every other city be $p = \$43$ while the cost of each of the remaining flights (connecting A_i and A_j, $i \neq 1, j \neq 1$) be $q = \$1/7$. Then the average of one flight is $2(99p + 99 \times 98q)/100 \times 99 = 1$ while the cost of any route including m cities which starts and ends at A_1 is $(2p + 97q)/99 > 99$.

(b) $m = 100$. *(Steven Chow)* In this case a route is a loop consisting of 100 cities. The routes differ only by the order of cities. No route passes through the same city twice. The traveller's home city is no special so he can start his route at any city.

Consider the total number of doubloons that the traveller must spent for every possible route. An average cost of one route is the sum of these costs divided by the number of possible routes.

Since every flight comes the same number of times in the total number of possible routes (it follows from symmetry of the situation), and since the average cost of one flight is 1 doubloon, the average cost of one route is 100 doubloons.

Then there is a route with the cost not exceeding 100 doubloons. Hence the traveller can always fulfil his plans. $\square$