INTERNATIONAL MATHEMATICS TOURNAMENT OF TOWNS

Senior O-Level, Spring 2014.

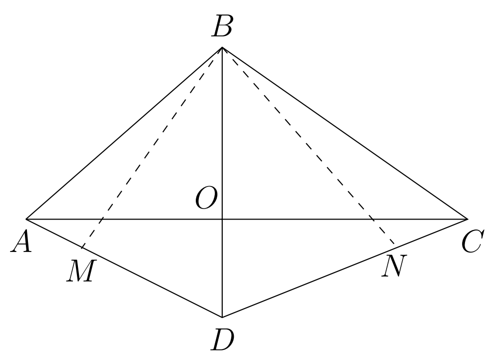
1. Inspector Gadget has 36 stones with masses 1 gram, 2 grams, ..., 36 grams. Doctor Claw has a superglue such that one drop of it glues two stones together (thus two drops glue 3 stones together and so on). Doctor Claw wants to glue some stones so that in obtained set Inspector Gadget cannot choose one or more stones with the total mass 37 grams. Find the least number of drops needed for Doctor Claw to fulfil his task.

Answer: 9

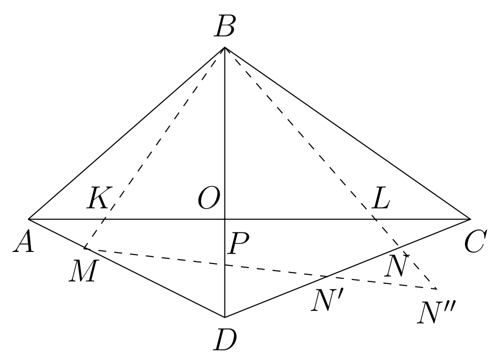
Solution. (a) Among the given stones there are 18 stones with odd masses which could be split into 9 pairs. To glue stones in pairs Doctor Claw needs 9 drops. In new group of stones there is no stone with odd weight. Therefore, Inspector Gadget cannot fulfil his task.

(b) Let us split all stones into 18 pairs so that in each pair a total weight of stones is 37. Then Doctor Claw needs to "spoil" at least one stone in each pair which is impossible with less than 9 drops.

2. In a convex quadrilateral $ABCD$ the diagonals are perpendicular. Points M and N are marked on sides AD and CD respectively. Prove that lines AC and MN are parallel given that angles ABN and CBM are right angles.



(a)



(b)

SOLUTION 1. See Figure (b). Observe that $\angle BAC = \angle OBL$ and $\angle KBO = \angle BCA$. Then triangles KBO and OBC are similar, so $KO : OB = OB : OC$ and therefore $KO = OB^2/OC$. In similar way, $OL = OB^2/OA$. Hence

$$\frac{KO}{OL} = \frac{AO}{OC}. \tag{$*$}$$

Assume that MN is not parallel to AC. Through M draw a line parallel to AC and denote points P, N' and N'' on it as shown. Then triangles ADC and MDN' are similar and therefore $MP : PN' = AO : OC$. Comparing to $(*)$ we conclude that

$$\frac{KO}{OL} = \frac{MP}{PN'}. \tag{$**$}$$

Since triangles MBN'' and KBL are also similar, we have $KO : OL = MP : PN''$. Comparing to $(*)$ we conclude that $PN' = PN''$. Contradiction.

SOLUTION 2. Introducing Cartesian coordinates one can assume that $A(a, 0)$, $B(0, b)$, $C(c, 0)$, $D(0, d)$ with $a < 0, b > 0, c > 0, d < 0$. Then MB is given by equation $cx - b(y - b) = 0$, and AD is given by equation $x/a + y/d = 1$. Solving the system we find y-coordinate of M: $y_M = (ac + b^2)d/(ac - bd)$. Permuting a and c we find $y_N = y_M$ which implies that $MN \parallel AC$.

3. Ali Baba and the 40 thieves want to cross Bosporus strait. They made a line so that any two people standing next to each other are friends. Ali Baba is the first; he is also a friend with the thief next to his neighbour. There is a single boat that can carry 2 or 3 people and these people must be friends. Can Ali Baba and the 40 thieves always cross the strait if a single person cannot sail?

SOLUTION. Let n be the number of the thieves (not counting Ali Baba). We will prove by induction that the gang can cross the strait. For $n = 1$ and 2 the base is obvious, one can check it for $n = 3$ as well. For simplicity of explanation we assume that they are going from Asia to Europe.

Assume that for any number $k = 1, 2, \ldots, n$ our statement holds. Denote Ali-Baba by A and the thieves by $T_1, \ldots, T_{n+1}$. First let $A, T_1, \ldots, T_{n-1}$ cross the strait leaving T_n, T_{n+1} behind in Asia (it can be done according to the induction hypotheses).

Next $A, T_1, \ldots, T_{n-2}$ sail back leaving T_{n-1} behind in Europe (again it can be done according to the induction hypotheses). Next T_n, T_{n+1} sail to Europe and then T_{n-1}, T_n go back bringing boat to Asia. Now $A, T_1, \dots, T_n$ are in Asia, so they cross the strait and join T_{n+1} .

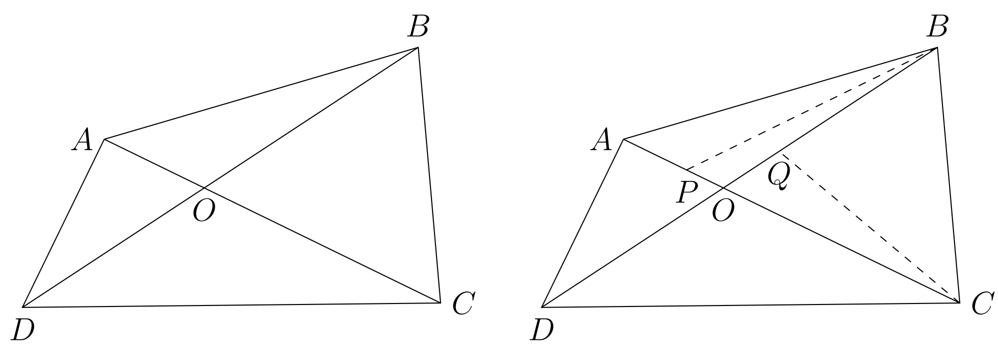
4. Positive integers a, b, c, d are pairwise coprime and satisfy the equation

$$ab + cd = ac - 10bd.$$

Prove that one can always choose three numbers among them such that one number equals the sum of two others.

Solution. Rewriting the equation in a form $a(c-b) = (10b+c)d$ and given that a and d are coprime we conclude that $(c-b)$ is positive and divisible by d: $c = b + dx$ with $x > 0$. Plugging c into latter equation and simplifying we get $(a-d)x = 11b$. Since c and b are coprime, x and b are also coprime and therefore either $x = 1$ or $x = 11$. In the former case we have $c = b + d$ and in the latter case $a = b + d$.

5. Park's paths go along sides and diagonals of the convex quadrilateral $ABCD$. Alex starts at A and hikes along $AB - BC - CD$. Ben hikes along AC; he leaves A simultaneously with Alex and arrives to C simultaneously with Alex. Chris hikes along BD; he leaves B at the same time as Alex passes B and arrives to D simultaneously with Alex. Can it happen that Ben and Chris arrive at point O of intersection of AC and BD at the same time? The speeds of the hikers are constant.



(a) (b)

Solution (Michael Chow, grade 12, Albert Campbell C.I.) Let v_A, v_B, v_C be the speeds of Alex, Ben and Chris respectively. Let Alex and Ben start hiking at time 0. By triangle inequality $AB + BC > AC$, so Alex traveled a further distance than Ben in the same time interval as they started and finished simultaneously. Hence $v_A > v_B$. Similarly $BC + CD > BD$ so Alex traveled a further distance than Chris in the same time interval. Hence $v_A > v_C$.

Assume that Ben and Chris arrive to O at the same time. Then we can replace them with a single person Mikey who travels from B to O with speed v_B and from O to C with speed v_C while Alex travels from B to C in the same time interval. Mikey's speed is always less than Alex's speed but Mikey travels distance $BO + OC$ which is greater than BC. This is impossible. Therefore, Ben and Chris cannot arrive at O at the same time.

Solution 2 (Frieda Rong, grade 11, Marc Garneau C.I.; Gloria Fang, grade 11, U.T.S). See Figure (b). Let P be a point where Ben was when both Alex and Chris were in B. Since speeds of Alex and Ben are constant and Ben arrive to C at the same time as Alex, we have $AB : BC = AP : PC$ and therefore BP is a bisector of $\angle ABC$.

Let Q be a point where Chris was when Alex and Ben arrived in C. Similarly $BQ : QD = BC : BD$ so CQ is a bisector of $\angle BCD$. Observe that Ben and Chris cannot arrive at O simultaneously unless P belongs to AO and Q belongs to OD.

Assume that Ben and Chris arrived to O simultaneously. Then $PO : OC = BO : OQ$ and since $\angle POB = \angle COQ$ we conclude that triangles BOB and COQ are similar. Then $\angle PBC + \angle BCQ = 180°$ and since BP and CQ are bisectors, $\angle ABC + \angle BCD = 360°$ which is impossible.