INTERNATIONAL MATHEMATICS TOURNAMENT OF TOWNS
Senior A-Level Paper, Spring 2014.

1. Doono wrote several **1**s, placed signs "+" or "×" between every two of them, put several brackets and got 2014 in the result. His friend Dunno replaced all "+" by "×" and all "×" by "+" and also got 2014. Can this be true?

SOLUTION. Yes, it could be true. For example, consider the following expression consisting of 4027 **1**s:

$$1+\underbrace{1\times 1+1\times 1+\ldots+1\times 1}_{\text{2013 terms}}$$

which obviously equals 2014. After Dunno changed signs it became

$$\underbrace{1\times 1+1\times 1+\ldots+1\times 1}_{\text{2013 terms}}+1$$

which also equals 2014.

2. Is it true that any convex polygon can be dissected by a straight line into two polygons with equal perimeters and

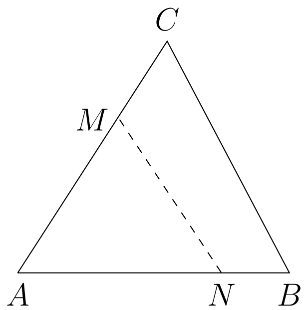
(a) equal greatest sides?

(b) equal smallest sides?

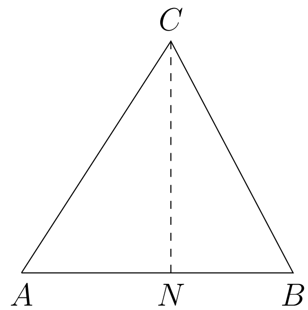
(a) ANSWER: Yes

SOLUTION. Consider a convex polygon polygon and point M on its boundary. Consider its opposite point $N = N(M)$. It means that MN dissects polygon into two $MA_1\ldots A_mN$ and $NA_{m+1}\ldots A_nM$ with equal perimeters (it is possible that M and N are among vertices of the original polygon). Here $MA_1\ldots A_mN$ is in the counterclockwise direction. Define $f(M)$ as a greatest side of $MA_1\ldots A_mN$. Observe that $f(M)$ continuously depends on M. Then $g(M) = f(M) - f(N(M))$ also continuously depends on M. However as M changes from original point M_0 to its opposite point N_0, $g(M)$ changes from $g(M_0)$ to $g(N_0) = -g(M_0)$. Therefore $g(M) = 0$ for some M.

Remark. $h(M)$ as the smallest side of $MA_1\ldots A_mN$ is not continuous and these arguments do not work for Part (b).



(a)



(b)

(b) ANSWER: No.

Consider triangle ABC. We call points M and N opposite if (as on the figure (a)) $MA + AN = p/2$ where $p = a + b + c$ is a perimeter of ABC.

Consider first an equilateral triangle with sides $a = b = c$. We claim that the smallest cut between opposite points has the length $3a/4$. Indeed, one can prove easily that the shortest cut MN must be orthogonal to bisector of angle CAB.

Therefore in an equilateral triangle $MN \geq 3a/4$ and $MC + NB = a/2$, $AM = NB + a/2$, $AN = CM + a/2$ and therefore for equilateral triangle the answer is negative unless the cut passes through one of the vertices. The same is true for all triangles sufficiently close to equilateral.

Consider $M = C$. But then in CAN and CNB the smallest sides are $AN = p/2 - b = (a - b + c)/2$ and $NB = (a + b - c)/2$ where $a = BC$, $b = AC$ and $c = AB$ and $AN \neq NB$ if $b \neq c$.

Therefore one cannot dissect any triangle which is close to equilateral but has all sides different.

3. The King called two wizards. He ordered First Wizard to write down 100 positive real numbers (not necessarily distinct) on cards without revealing them to Second Wizard. Second Wizard must correctly determine all these numbers, otherwise both wizards will lose their heads. First Wizard is allowed to provide Second Wizard with a list of distinct numbers, each of which is either one of the numbers on the cards or a sum of some of these numbers. He is not allowed to tell which numbers are on the cards and which numbers are their sums. Finally the King tears as many hairs from each wizard's beard as the number of numbers in the list given to Second Wizard. What is the minimal number of hairs each wizard should lose to stay alive?

ANSWER. 101

SOLUTION [Coincides with given by Ben Wei]. The first wizard writes $1, 2, 4, \ldots, 2^{99}$ and lists all these numbers and their sum $2^{100} - 1$. Then the second wizard understands that there is a card with a number not exceeding 1, there is another card with a number not exceeding 2, ..., and there is 100th card with a number not exceeding 2^{99}. Then their sum does not exceed $2^{100} - 1$ and the emuality is possible if and only if numbers are $1, 2, 4, \ldots, 2^{99}$.

4. In the plane are marked all points with integer coordinates (x, y), $0 \leq y \leq 10$. Consider a polynomial of degree 20 with integer coefficients. Find the maximal possible number of marked points which can lie on its graph.

SOLUTION (Michael Chow) (i) We need to consider integer solutions of the system of inequalities:

$$0 \leq P(x) \leq 10. \qquad (*)$$

Let us prove by contradiction that there are no more than 20 integer solutions to $(*)$. Assume that $x_1 < x_2 < \ldots < x_{21}$ satisfy $(*)$; denote $a = x_1$, $b = x_{21}$; then $b - a \geq 20$.

Consider $P(b) - P(a)$; since both a, b satisfy $(*)$ we conclude that $|P(b) - P(a)| \leq 10$. However since $P(x)$ has integer coefficient, the number $P(b) - P(a)$ must be divisible by $(b - a)$ (indeed, $P(b) - P(a) = (b - a)R(a, b)$ where R is a polynomial with integer coefficients). Since $|P(b) - P(a)| \leq 10$ and $b - a \geq 20$ divisibility implies that $P(b) - P(a) = 0$. So $P(a) = P(b) = c$ with $0 \leq c \leq 10$.

Then $P(x) = (x - a)(b - x)Q(x) + c$ where $Q(x)$ is a polynomial of degree 18. Observe that $(x - a)(b - x) \geq 19$ for integer $x = a + 1, \ldots, b - 1$. Then $P(x)$ cannot satisfy $(*)$ unless $Q(x) = 0$. Indeed, if $Q(x) \neq 0$ then either $P(x) \leq -19 + c < 0$ or $P(x) \geq 19 + c > 10$.

Therefore $Q(x_k) = 0$, $k = 2, \ldots, 20$ but polynomial $Q(x)$ of degree 18 cannot have more than 18 roots. Contradiction.

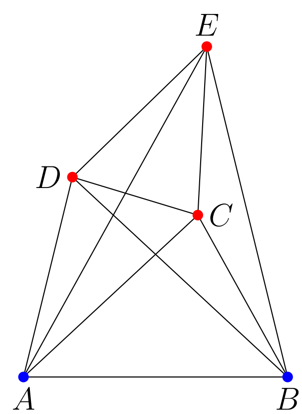
(ii) On the other hand, for $P(x) = (x - x_1)(x - x_2)\cdots(x - x_{20})$ $(*)$ has 20 solutions $x_1, \ldots, x_{20}$.

SOLUTION 2. We need to consider integer solutions of the system of inequalities $(*)$. Let us prove by contradiction that there are no more than 20 integer solutions to $(*)$. Assume that $x_1 < x_2 < \ldots < x_{21}$ satisfy $(*)$. By Bézout's theorem $P(x_{21} - P(x_1)$ is divisible by $x_{21} - x_1 \geq 20$ and therefore $P(x_{21} = P(x_1) = r$. Similarly $P(x_{21}) = P(x_i) = r$ for all $i = 2, \ldots, 10$ since $x_{21} - x_i \geq 11$. Also $P(x_1) = P(x_k) = r$ for all $k = 12, \ldots, 21$ since $x_k - x_1 \geq 11$. Therefore all x_j except x_{11} are roots of $P(x) - r$ and thus $P(x) = a(x - x_1) \cdots (x - x_{10})(x - x_{12}) \cdots (x - x_{21}) + r$. But then $|P(x_{12}) - r| \geq (10!)^2$ which is a contradiction.

5. There is a scalenetriangle. Peter and Basil play the following game. On each his turn Peter chooses a point in the plane. Basil responds by painting it into red or blue. Peter wins if some triangle similar to the original one has all vertices of the same colour. Find the minimal number of moves Peter needs to win no matter how Basil would play (independently of the shape of the given triangle)?

Answer. 5

Solution. Peter selects triangle ABC (an original one). Basil paints A and B blue and C red. Then Peter selects D and E on the same side of AB as C so that triangles ABC, BDA and EAB are similar (with vertices in the matching order). Basil is forced to paint them red.



Now prove that triangle EDC and EAB are similar. Observe that $\angle DAE = \angle CBE$. Indeed, $\angle DAE = \angle DAB - \angle EAD$ and $\angle CBE = \angle ABE - \angle ABC$ and those angles are equal due to similarity. Also $DA : BC = AB : CA = EA : BE$.

Thus triangles DAE and CBE are similar and in triangles EDC and EAB angles E are equal and $DE : CE = AE : BE$ and therefore they are similar.

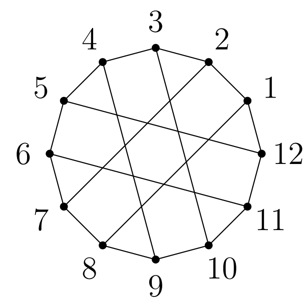
Remark. This could be described using complex numbers terminology. Indeed, let us introduce coordinate system on the complex plane $\mathbb{C}$ such that points C, A and B correspond to complex numbers 0, z and z^2 respectively (it is always possible). Let us add points $w(E)$ $wz(D)$ with $w = z^2 - z + 1$. Then triangles CAB, ABD, BEA CED are similar. Indeed triangles CAB and CED could be obtained from triangle Δ with vertices $(0, 1, z)$ by multiplication by z and w respectively; triangle BEA could be obtained from Δ by multiplication by $(1-z)$ and shift by z^2, and triangle ABD could be obtained from Δ by multiplication by z^2-z and shift by z.

6. In some country every town has a unique number. In a flight directory for any two towns there is an indication whether or not they are connected by a direct non-stop flight. It is known that for any two assigned numbers M and N one can change the numeration of towns so that the town with number M gets the number N but the directory remains correct.

Is it always true that for any two assigned numbers M and N one can change the numeration of towns so that the towns with numbers M and N interchange their numbers but the directory is still correct?

Answer: No

Solution. Observe that figure is symmetric with respect to each diameter passing through the middle of the small chord. These symmetries allows us to interchange neighbouring towns and then several symmetries allows us to transfer any town into any other town.



Assume that we can exchange towns 1 and 3. Then their only common connected town 2 must remain on its place. Then its connected town 9 must also remain on its place. But 3 and 9 have two common connected towns (8 and 2) while 1 and 9 have only one common connected town (2).

Remark. Another example: tetrahedron with cut vertices. Then there is a graph with 12 vertices and 18 edges which has the same properties.

7. Consider a polynomial $P(x)$ such that

$P(0)=1; \quad (P(x))^2 = 1 + x + x^{100}Q(x),$ where $Q(x)$ is also a polynomial.

Prove that in the polynomial $(P(x)+1)^{100}$ the coefficient at x^{99} is zero.

Solution. Observe that $(P(x)+1)^{100} + (1-P(x))^{100}$ contains only even powers of $P(x)$ and therefore is a polynomial of degree 50 of $(P(x))^2$ i.e. of $(1+x)^{50}$ modulo polynomial divisible by x^100. However $1-P(x)$ is divisible by x and therefore $(1-P(x))^{100}$ is divisible by x^{100}.

Remark. More generally $(P(x)+1)^n$ is a polynomial of degree $\lfloor n/2 \rfloor$ modulo polynomial divisible by x^n and therefore coefficients at x^m, $m = \lfloor n/2 \rfloor + 1, \dots, n-1$ are zeros.