INTERNATIONAL MATHEMATICS TOURNAMENT OF TOWNS

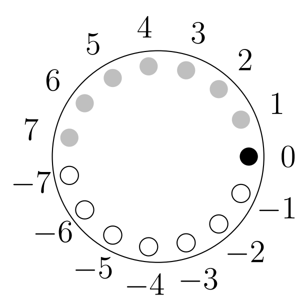
Junior O-Level, Spring 2014.

1. Each of given 100 numbers was increased by 1. Then each number was increased by 1 once more. Given that the first time the sum of the squares of the numbers was not changed find how this sum was changed the second time.

SOLUTION. Given that the sum of the squares did not change when we added 1 to each number, we have $(a_1+1)^2+(a_2+1)^2+\cdots+(a_{100}+1)^2)-(a_1^2+a_2^2+\cdots+a_{100}^2)=0$ or $(2a_1+1)+(2a_2+1)+\cdots+(2a_{100}+1)=0$. Therefore, we have $a_1+a_2+\cdots+a_{100}=-50$. If we increase each number by 1 once more, the sum of squares will be change by $(a_1+2)^2+(a_2+2)^2+\cdots+(a_{100}+2)^2-(a_1^2+a_2^2+\cdots+a_{100}^2)=(4a_1+4)+(4a_2+4)+\cdots+(4a_{100}+4)=4\times(-50)+400=200$.

2. Mother baked 15 pasties. She placed them on a round plate in a circular way: 7 with cabbage, 7 with meat and one with cherries in that exact order and put the plate into a microwave. All pasties look the same but Olga knows the order. However she doesn't know how the plate has been rotated in the microwave. She wants to eat a pasty with cherries. Can Olga eat her favourite pasty for sure if she is not allowed to try more than three other pasties?

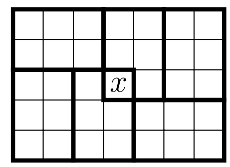
SOLUTION. Denote the cherry pasty by 0, the cabbage pasties by 1,..., 7 and the meat pasties by $-1,\dots,-7$. If Olga does not get the cherry pasty on her first try, it must be either a cabbage pasty or a meat pasty. On her second try Olga takes the 4-th pasty from the first one in the direction of the cherry pasty. She gets either the cherry pasty 0, or the cabbage pasty 1,2,3, or the meat pasty $-1,-2,-3$.



On her last try Olga takes the second pasty from her second try in the direction of the cherry pasty and gets either the cherry pasty 0, or the cabbage pasty 1, or the meat pasty -1. Hence, after at most three tries Olga knows the position of the cherry pasty for sure.

3. The entries of a 7×5 table are filled with numbers so that in each 2×3 rectangle (vertical or horizontal) the sum of numbers is 0. For 100 dollars

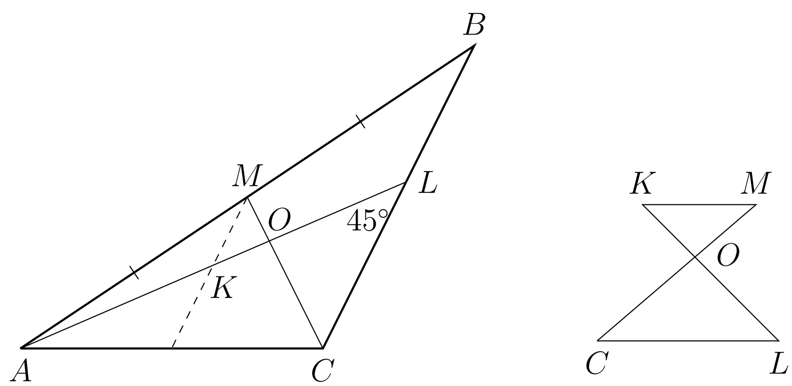
Peter may choose any single entry and learn the number in it. What is the least amount of dollars he should spend in order to learn the total sum of numbers in the table for sure?



Solution. Let S be a total sum of the numbers in the table. Let Peter divide the table into 6 rectangles as shown on the picture (two rectangles overlap on a marked entry). Then $S = 0 \times 5 + (0 - x)$ where x is the value in the marked entry he would pays for. Since a 7×5 table cannot be split into 2×3-rectangles without overlapping (or holes) Peter cannot find S for free.

4. Point L is marked on side BC of triangle ABC so that AL is twice as long as the median CM. Given that angle ALC is equal to $45°$ prove that AL is perpendicular to CM.

Solution.



(a) (b)

Let O be a point of intersection of KL and MC. Let K be a point of intersection of AL and a line drawn through M parallel to BC. Then $AK = KL$. Since MK is parallel to CL, triangles KMO and OLC are similar and we have $KO/(KL - KO) = MO/(MC - OM)$. Since $KL = MC$, $KO = OM$ and each of triangles OKM and OCL is isosceles and therefore $\angle OCL = \angle OLC = 45°$. Hence, $\angle COL = 90°$.

5. Ali Baba and the 40 thieves want to cross Bosporus strait. They made a line so that any two people standing next to each other are friends. Ali Baba is the first; he is also a friend with the thief next to his neighbour. There is a single boat that can carry 2 or 3 people and these people must be friends. Can Ali Baba and the 40 thieves always cross the strait if a single person cannot sail?

SOLUTION. (Vassily Kapustin, grade 8, Poplar Bank P.S.) If the number of thieves n is even then A, T_1, T_2 sail to Europe, and A, T_1 sail back leaving T_2 in Europe. Then T_3, T_4 sail to Europe, T_2, T_3 sail back and now T_4 is in Europe and everybody else is in Asia. Continuing this process we end up with T_n (the last thief in line) in Europe, and $A, T_1, \ldots, T_{n-1}$ with the boat in Asia.

If the number of thieves n is odd then A, T_1, T_2 sail to Europe, and A, T_2 sail back leaving T_1 in Europe. Then T_2, T_3 sail to Europe, T_1, T_2 sail back and now T_3 is in Europe and everybody else is in Asia. Continuing this process we end up with T_n in Europe, and $A, T_1, \ldots, T_{n-1}$ with the boat in Asia.

We can see that after applying described operation the last thief in the line will be in Europe and all the remained gang in Asia. We are again in conditions of the original problem but the number of thieves decreased by 1. Therefore, we apply this process several times until we get only A, T_1, T_2 in Asia. Then the trio sail to Europe and join the gang.