INTERNATIONAL MATHEMATICS TOURNAMENT OF TOWNS
Junior A-Level Paper, Spring 2014.

1. During Christmas party Santa handed out to the children 47 chocolates and 74 marmalades. Each girl got 1 more chocolate than each boy but each boy got 1 more marmalade than each girl. What was the number of the children?

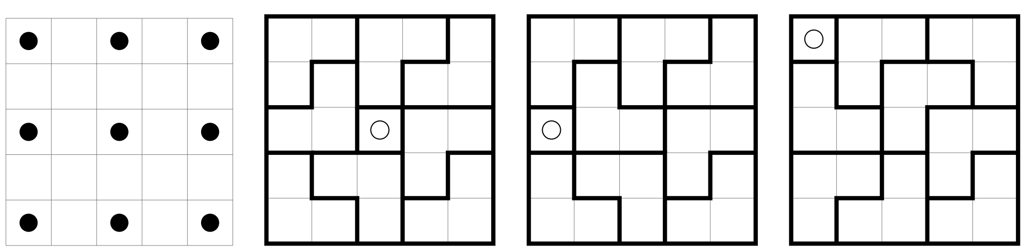
SOLUTION. Each child got the same number of treats and the total number of treats is $74 + 47 = 121$. Therefore there could be either (a) 11 children, or (b) 121, or (c) just 1 child, and each child got 11, 1, or 121 treat respectively.

Remark. In case (a) let x denote the number of boys and c the number of chocolates each girl got. Then $(c-1)x + c(11-x) = 47$ or $11c = 47 + x$. The only integer solution with $0 \le x \le 11$ is $x = 8$, $c = 5$ (so, 8 boys, 3 girls). In case (b) each boy got just 1 marmalade, and each girl got just 1 chocolate (so, 74 boys and 47 girls). Case (c) is correct from the point of view of formal logic.

2. Peter marks several cells on a 5×5 board. Basil wins if he can cover all marked cells with three-cell corners. The corners must be inside the board and not overlap. What is the least number of cells Peter should mark to prevent Basil from winning? (Cells of the corners must coincide with the cells of the board).

SOLUTION. If Peter marks 9 points as shown on (a) Basil cannot cover them. Indeed, no corner can cover more than one marked cell, so Basil needs 9 corners; but they contain 27 cells while the whole board contains only 25.

If Peter marks 8 cells Basil can cover all of them. Indeed, one of the cells shown on (a) is not marked. However the remaining 24 cells could be covered as shown on (b)–(d).

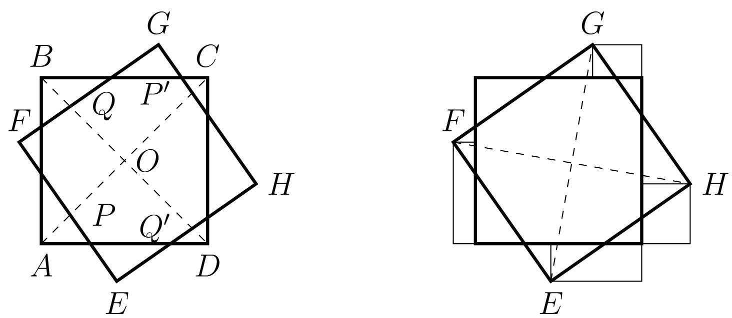


(a) (b) (c) (d)

3. A square table is covered with a square cloth (may be of a different size) without folds and wrinkles. All corners of the table are left uncovered and all four hanging parts are triangular. Given that two adjacent hanging parts are equal prove that two other parts are also equal.

Solution 1. Let $ABCD$ be a cloth and $EFGH$ be a table (see Figure (a)). We see four hanging parts of the cloth and four triangular parts of the table which are not covered. Observe that all eight triangles are similar. Let us draw diagonals in $ABCD$. Observe that they are bisectors of the corresponding angles. Observe that since angles between AC and $FG \parallel EH$ and BD and $FE \parallel EH$ are equal and distances between two pairs of parallel lines are also equal then $QQ' = PP'$.

If triangles A and B are equal then their bisectors AP and BQ are equal and since $AO = BO = CO = DO$ we see that $PO = QO$. But then $P'O = Q'O$ and $P'C = Q'D$. Then triangles C and D are also equal.



(a) (b)

Solution 2 (see Figure (b)). We define the *weight* of the hanging triangle as its height dropped from the right corner. Obviously all hanging parts are similar. Therefore parts are equal if and only if their heights are equal. Therefore it is sufficient to prove that the the sums of wights of opposite parts are equal. Adding to these sums the side of the table we get projection of diagonal FH to the "horizontal" side of the the table and of diagonal EG to the "vertical" side of the the table. Since diagonals are equal and orthogonal and the sides of the table are orthogonal, we conclude that projections are equal.

4. The King called two wizards. He ordered First Wizard to write down 100 positive integers (not necessarily distinct) on cards without revealing them to Second Wizard. Second Wizard must correctly determine all these integers, otherwise both wizards will lose their heads. First Wizard is allowed to provide Second Wizard with a list of distinct integers, each of which is either one of the integers on the cards or a sum of some of these integers. He is not allowed to tell which integers are on the cards and which integers are their sums. Finally the King tears as many hairs from each wizard's beard as the number of integers in the list given to Second Wizard. What is the minimal number of hairs each wizard should lose to stay alive?

Solution [Coincides with given by Ben Wei]. The first wizard writes $1, 2, 4, \dots, 2^{99}$ and lists all these numbers and their sum $2^{100} - 1$. Then the second wizard understands that there is a card with a number not exceeding 1, there is another card with a number not exceeding 2, $\dots$, and there is 100th card with a number not exceeding 2^{99}. Then their sum does not exceed $2^{100} - 1$ and the emuality is possible if and only if numbers are $1, 2, 4, \dots, 2^{99}$.

5. There are several white and black points. Every white point is connected with every black point by a segment. Each segment is equipped with a positive integer. For any closed circuit the product of the numbers on the segments passed in the direction from white to black point is equal to the product of the numbers on the segments passed in the opposite direction. Can one always place the numbers at each point so that the number on each segment is the product of the numbers at its ends?

Solution. Let us denote white points W_j, $j = 1, 2, \dots, m$ and black points B_k, $k = 1, 2, \dots .n$. Let c_{jk} be a label on the segment from white point W_j to black point B_k. Consider closed circuit $W_1 - B_1 - W_j - B_k - W_1$. Then $c_{11}c_{jk} = c_{j1}c_{1k}$ and therefore $c_{jk} = c_{j1}c_{1k}/c_{11} = w_j b_k$ where $w_j = c_{j1}/d$, $b_k = c_{1k}d/c_{11}$, $d = \gcd(c_{11}, c_{21}, \dots, c_{m1})$. Obviously $w_1, \dots, w_m$ are integers and coprime. Since $w_j b_k$ are integers and $w_1, \dots, w_m$ are coprime, then b_k are integers as well.

Indeed, let b_k be not an integer, then represent it as irreducible ratio $b_k = b'/r$ with $r \geq 2$. Since $w_j b_k = w_j b'/r$ are integers r must divide w_j ofr all j which is impossible as these numbers are coprime.

6. A $3 \times 3 \times 3$ cube is made of $1 \times 1 \times 1$ cubes glued together. What is the maximal number of small cubes one can remove so the remaining solid has the following features:

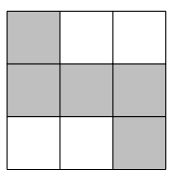
1) Projection of this solid on each face of the original cube is a 3×3 square;

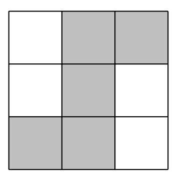
2) The resulting solid remains face-connected (from each small cube one can reach any other small cube along a chain of consecutive cubes with common faces).

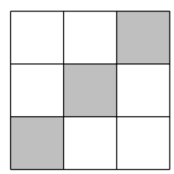
Answer: 14 small cubes.

Solution. Consider example with removed 14 cubes (remaining 13 cubes are shaded on these 3 layers). Each layer has cubes in each row and column; imposing layers we get a full square. Therefore the first condition is fulfilled. Top and middle layers are glued together through their central cubes. Each cube of the bottom layer is glued to the corresponding cube of the middle layer.

To prove that no more than 14 cubes could be removed we prove an estimate for the number n of remaining cubes. We can see from all 6 directions $6 \cdot 9$ of their faces. To connectivity one needs at least $(n-1)$ gluings ; therefore we do not see at least $2(n-1)$ faces, whole the total number of faces is $6n$. Then $6n \geq 2(n-1)+54$ and therefore $n \geq 13$.


(a) Top

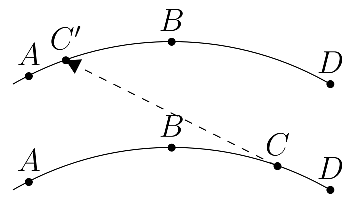

(b) Middle


(c) Bottom

7. Points A_1, A_2, …, A_{10} are marked on a circle clockwise. It is known that these points can be divided into pairs of points symmetric with respect to the centre of the circle. Initially at each marked point there was a grasshopper. Every minute one of the grasshoppers jumps over its neighbour along the circle so that the resulting distance between them doesn't change. It is not allowed to jump over any other grasshopper and to land at a point already occupied. It occurred that at some moment nine grasshoppers were found at points A_1, A_2, …, A_9 and the tenth grasshopper was on arc $A_9A_{10}A_1$. Is it necessarily true that this grasshopper was exactly at point A_{10}?

Answer. Yes.

Solution. 10 grasshoppers divide circle into 10 arcs. Let us paint alternatively black and white. Originally sums of the lengths of white and black arcs are equal because for any white arc an arc which is symmetric to it with respect to the center is black and conversely for any black arc an arc which is symmetric to it with respect to the center is white.



It follows from the figure that the grasshopper's jump does not change these sums. Indeed, sum of arcs AC' and BD equals to the sum of arcs AB and CD. In the final configuration we know 4 black arcs and know where the fifth is located and therefore position of 10th grasshopper is defined uniquely. On the other hand, A_{10} satisfies to the sums of black and white arcs are equal condition.