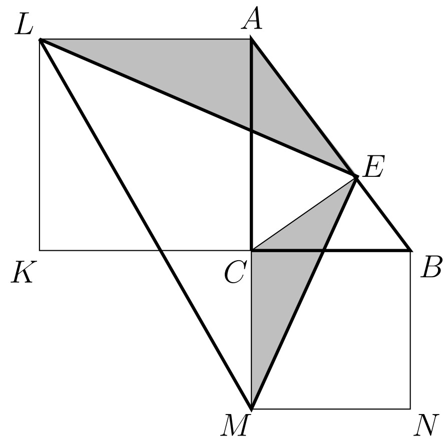
International Mathematics
TOURNAMENT OF THE TOWNS SOLUTIONS

Senior O-Level Paper **Spring 2013.**

1 [**3**] There is a positive integer A. Two operations are allowed: increasing this number by 9 and deleting a digit equal to 1 from any position. Is it always possible to obtain $A+1$ by applying these operations several times?

SOLUTION. See Junior 1.

2 [**4**] Let C be a right angle in triangle ABC. On legs AC and BC the squares $ACKL$, $BCMN$ are constructed outside of triangle. If CE is an altitude of the triangle prove that LEM is a right angle.



SOLUTION 1. Since ABC is right triangle and CE is perpendicular to AB, triangles CBE and ACE are similar. Then we have

(a) $\angle CAB = \angle ECB$ (and therefore, $\angle LAE = \angle MCE$) and also

(b) $CM/CE = AL/AE$ (it follows from $CB/CE = AC/AE$).

Therefore, triangles ALE and CME are similar. Then $\angle ALE = \angle EMC$ and therefore quadrilateral $LAEM$ is cyclic. This implies $\angle LEM = \angle LAM = 90°$.

SOLUTION 2. It is easy to see that AEC and ACB are similar, hence $\frac{CE}{EA} = \frac{CB}{CA} = \frac{CM}{AL}$. Thus the rotation by 90° followed by homothety with center E and factor $\frac{CE}{CA}$ transforms segment EA into segment EC and line AL into line CM. Then segment AL transforms into CM while segment EL into segment EM. Hence $\angle LEM = 90°$.

3 [**4**] Eight rooks are placed on a 8×8 chessboard, so no two rooks attack one another. All squares of the board are divided between the rooks as follows. A square where a rook is placed belongs to it. If a square is attacked by two rooks then it belongs to the nearest rook; in case these two rooks are equidistant from this square then each of them possesses a half of the square. Prove that every rook possesses the equal area.

SOLUTION. See Junior 4.

4 [**4**] Each of 100 stones has a sticker showing its true weight. No two stones weight the same. Mischievous Greg wants to rearrange stickers so that the sum of the numbers on the stickers for any group containing from 1 to 99 stones is different from the true weight of this group. Is it always possible?

ANSWER: Yes.

SOLUTION. Let us arrange the stones in a circle in the increasing order of their weights clockwise. Let Greg move a sticker from each stone to the next one in the counterclockwise direction. In this way the heaviest stone will get a sticker with the smallest number while every other stone gets a sticker with the number greater than its true weight. Therefore for any group of stones which does not include the heaviest stone the sum of the numbers on the stickers will be greater than the total true weight of stones of this group. On the other hand, if a group contains the heaviest stone (but not all stones), then the complementary group (that is, all other stones) does not contain it; therefore in that group the sum of the numbers on the stickers will be greater than the true total weight of stones. Then in the chosen group the sum of the numbers on the stickers will be smaller than the true weight of stones of this group. Hence, Greg can fulfill his task.

5 [**5**] A quadratic trinomial with integer coefficients is called *admissible* if its leading coefficient is 1, its roots are integers and the absolute values of coefficients do not exceed 2013. Basil has summed up all admissible quadratic trinomials. Prove that the resulting trinomial has no real roots.

SOLUTION. Consider admissible trinomial $x^2 + Bx + C$ and note that the polynomial $x^2 - Bx + C$ is also admissible. Then the sum of all admissible polynomials is $Ax^2 + C$ with $A > 0$. We need to prove that $C > 0$.

For each pair (a, b) of integers such that $0 \leq a \leq b \leq 2013$, $ab \leq 2013$, consider all admissible trinomials with roots (a, b), $(-a, b)$, $(-a, b)$, $(-a, -b)$. Consider the following cases:

(1) $a = 0$; then the contribution to C is 0.

(2) $a = 1$, $b = 2013$. In this case, there are two admissible trinomials, $x^2 \pm 2012x - 2013$; their joint contribution to C equals -4026.

(3) $a = 1 < b < 2013$. Then there are four admissible trinomials, $x^2 \pm (b + 1)x + b$. $x^2 \pm (b - 1)x - b$; their joint contribution to C equals 0.

(4) $1 < a < b \leq 2013$. Then since $ab \leq 2013$ we have $a < b < 2013/2$ and therefore $a + b < 2013$. Thus again we have four trinomials; their joint contribution to C equals 0.

(5) $1 \leq a = b$, $a^2 < 2013$. We get three trinomials: $x^2 \pm 2a + a^2$, and $x^2 - a^2$; their joint contribution to C equals a^2. In total, $C = 1^2+2^2+\ldots+44^2-4026 = \frac{1}{6}\cdot 44\cdot 45\cdot 89-4026 > 0$.