International Mathematics
TOURNAMENT OF THE TOWNS SOLUTIONS

Junior O-Level Paper **Spring 2013.**

1 [3] There are six points on the plane such that one can split them into two triples each creating a triangle. Is it always possible to split these points into two triples creating two triangles with no common point (neither inside, nor on the boundary)?

ANSWER: No.

Example: Consider the vertices and the midpoints of a triangle.

2 [4] There is a positive integer A. Two operations are allowed: increasing this number by 9 and deleting a digit equal to 1 from any position. Is it always possible to obtain $A+1$ by applying these operations several times?

REMARK. If leading digit 1 is deleted, all leading zeros are deleted as well.

ANSWER: Yes.

SOLUTION. Given the number $A+1$ create a "new number" which starts with eight "1"s followed by the number $A+1$. Note that the new number and the number A have the same remainders when divided by 9. Therefore given the number A one can get the number $A+1$ by adding "9"s to A until one obtains the "new number". Then one removes eight leading "1"s.

3 [4] Each of 11 weights is weighing an integer number of grams. No two weights are equal. It is known that if all these weights or any group of them are placed on a balance then the side with a larger number of weights is always heavier. Prove that at least one weight is heavier than 35 grams.

SOLUTION. Let us arrange the weights in increasing order, $a_1 < a_2 < a_3 \cdots < a_{11}$. Note that the difference between any two consequent weights is at least 1. Therefore, $a_n \geq a_m + (n-m)$, if $m < n$. According to the given we have

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_2 + a_6 > a_7 + a_8 + a_9 + a_{10} + a_{11}$$

and since

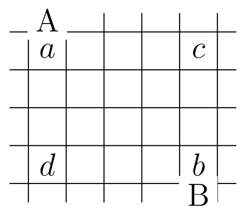
$$a_7+a_8+a_9+a_{10}+a_{11} \geq (a_2+5)+(a_3+5)+\ldots+(a_6+5) = a_2+a_3+a_4+a_5+a_6+25$$

we have $a_1 > 25$. Then $a_{11} \geq a_1 + 10 > 35$.

4 [5] Eight rooks are placed on a 8×8 chessboard, so that no two rooks attack one another. All squares of the board are divided between the rooks as follows. A square where a rook is placed belongs to it. If a square is attacked by two rooks then it belongs to the nearest rook; in case these two rooks are equidistant from this square each of them possesses a half of the square. Prove that every rook possesses the equal area.

SOLUTION. Observe that a rook attacks 15 squares in total, 7 squares in a column and 7 squares in a row where it stands plus a square where it stands.

Let us denote a square and rook that stands on it by the same letter, correspondingly small and capital. Let rook A stand on square a. Consider a square c in the same row with square a. It is attacked by another rook B which stands on square b in the same column with c. Rook B will also attack a square d which is in the same column with a. The squares a, b, c, d are the corners of a rectangle. If it is a square then each rook A and B gets a half of c and d. Otherwise, one of the squares completely belongs to rook A, and another to rook B. Consequently, each rook possesses 8 squares in total: the square it stands on and a half of the remaining 14 squares. The statement holds for every rook.



5 [5] In a quadrilateral $ABCD$, angle B is equal to $150°$, angle C is right, and sides AB and CD are equal. Determine the angle between BC and the line connecting the midpoints of sides BC and AD.

ANSWER. 60 . SOLUTION. Let M be the midpoint of BC and N be the midpoint of AD. Construct parallelogram $ABMK$ and rectangle $CDLM$. Observe that $AKDL$ is also a parallelogram. (Indeed, $AK = LD$ and AK is parallel to LD). Hence N is the midpoint of diagonal KL. In triangle KML, $\angle KML = \angle KMC - \angle LMC = 150° - 90° = 60°$. Since $KM = ML$, triangle KML is equilateral. Then the median MN of triangle KML is also a bisector and therefore $\angle KMN = 30°$ and $\angle BMN = 60°$.