International Mathematics
TOURNAMENT OF THE TOWNS SOLUTIONS

Junior A-Level Paper **Spring 2013.**

1 [**4**] Several positive integers are written on a blackboard. The sum of any two of them is some power of two (for example, $2, 4, 8, \ldots$). What is the maximal possible number of different integers on the blackboard?

ANSWER: Two.

SOLUTION 1. Let a be the greatest number written on a blackboard. There is an integer $n \geq 0$ such that $2^n \leq a < 2^{n+1}$. Then $2^n < a+b \leq 2a < 2^{n+2}$ where b is the other number on the board. Hence $a+b = 2^{n+1}$. Thus all the remaining integers are in the form $2^{n+1} - a$. Therefore the number of different integers on the board is no more than two.

Example of two integers: 1 and 3.

SOLUTION 2. We prove that the number of integers does not exceed 2. Assume that $a < b < c$ on the board. Then $a+b < a+c < b+c$ are different powers of 2 and therefore $b+c \geq 2(a+c)$. Then $b \geq 2a + c$ which is impossible. Example of two integers: 1 and 3.

2 [**4**] Twenty children, ten boys and ten girls, are standing in a line. Each boy counted the number of children standing to the right of him. Each girl counted the number of children standing to the left of her. Prove that the sums of numbers counted by the boys and the girls are the same.

SOLUTION 1. Assume that the children in a line stay to the right of the first person. Let a boy on the k-th position count the number $20 - k$ while a girl on the n-th position count the number $n - 1$. Therefore the total sums of numbers obtained by boys and girls are $200 - S_b$ and $S_g - 10$ respectively, where S_b is the sum of boys' positions and S_g is the sum of girls' positions. It remains to check that $200 - S_b = S_g - 10$. The latter follows from $S_b + S_g = 1 + 2 + \cdots + 20 = 210$.

SOLUTION 2. Let B and G be the sums counted by boys and girls respectively. Note that if a boy and a girl interchange their places in the line, both sums will increase or decrease on the same amount. Therefore the difference between B and G is always the same. However, in situation when ten girls are followed by ten boys it is obvious that both sums are the same.

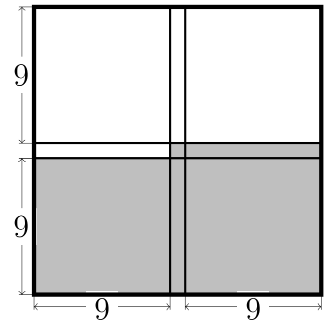
3 [**5**] There is a 19×19 board. Is it possible to mark some 1×1 squares so that each of 10×10 squares contain different number of marked squares?

ANSWER. Yes, it is possible. SOLUTION. Observe that each of 100 of 10×10 squares in a 19×19 shares a common central cell (1×1 square). Assume that it is marked; otherwise, we can interchange marked and unmarked cells.

Let us mark every cell in each of nine bottom rows, the central cell and all cells in central row to the right of it. Consider a 10×10 square at the top left position. It has one marked cell. Let us move this square to the right, one column at time. In this way, each new 10×10 square will have one more marked cell than the previous one. Therefore we get squares with 1,2,. . . ,10 marked cells.

Now, move each of these ten squares down one row at time. It is easy to see that each new 10×10 square contains 10 more marked cells than the square one position above it. In this way, we get squares with 11, 21, 31,. . . ,91, 12, 22, 32,. . . ,92,. . . , 20, 30,. . . , 100 marked cells.

4 [**5**] On a circle, there are 1000 nonzero real numbers painted black and white in turn. Each black number is equal to the sum of two white numbers adjacent to it, and each white number is



equal to the product of two black numbers adjacent to it. What are the possible values of the total sum of 1000 numbers?

Solution 1. Let w_n be the n-th white number, b_n be the n-th black number and S_w and S_b be the total sums of white and black numbers respectively (we assume that w_1 is left neighbour of b_1, $w_{n+500} = w_n$ and $b_{n+500} = b_n$). Then

$$b_n = w_n + w_{n+1} \qquad \text{and} \qquad w_n = b_{n-1}b_n. \tag{1}$$

Then (1) implies $b_n = b_{n-1}b_n + b_nb_{n+1}$ and since $b_n \neq 0$ we have $b_{n-1} + b_{n+1} = 1$. Then $S_b = b_1 + b_2 + \ldots + b_{500} = 250$ as we split $b_1, \ldots, b_{500}$ into 250 pairs (b_{n-1}, b_{n+1}).

On the other hand, $S_b = (w_1 + w_2) + (w_2 + w_3) + \ldots + (w_{499} + w_{500}) + (w_{500} + w_1) = 2S_w$, therefore $S_w = 250/2 = 125$. Finally, the total sum of all numbers is $S_w + S_b = 125 + 250 = 375$.

Solution 2. Let a be a value of some black number. Assume that the value of neighbouring white number is ab. Then the following six numbers are uniquely determined: b, $b - ab$, $1 - a$, $(1 - a)(1 - b)$, $1 - b$, $a(1 - b)$. It is easy to check that the sum of these eight numbers is 3. Since the given 1000 numbers can be split into 125 consecutive groups, the total sum of all numbers is $3 \times 125 = 375$.

5 [**6**] A point in the plane is called a node if both its coordinates are integers. Consider a triangle with vertices at nodes containing exactly two nodes inside. Prove that the straight line connecting these nodes either passes through a vertex or is parallel to a side of the triangle.

Solution 1

Lemma. Suppose that X and Y are interior points of triangle ABC and the segment XY is not parallel to any side of the triangle. Then there exists a segment equal and parallel to XY, with one endpoint at a vertex and the other endpoint inside the triangle.

Proof. Through each vertex of triangle ABC draw a line parallel to XY. Consider the line that is between two others. Assume it passes through vertex A. By D denote the point of intersection of this line with side BC. Since segment XY is parallel to AD and lies completely in the interior of the triangle, XY is shorter than AD. Mark a point Z on AD so that $AZ = XY$. By construction, point Z is an interior point of triangle ABC. □

Let us proceed with the problem. Let X and Y be two nodes inside triangle ABC. If line XY is parallel to any side of the triangle, the statement holds. Otherwise, in accordance with the lemma we construct point Z. Note that Z is a node. By condition Z coincides with either X or Y. Therefore X and Y belong to the line passing through a vertex of a triangle.

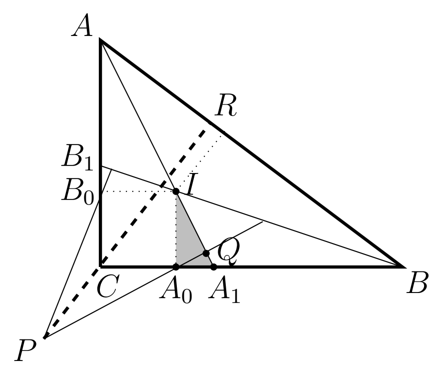
Solution 2 (for advanced participants). Assume that line XY does not pass through any vertex of triangle ABC. Then there is a side (let it be BC) which XY does not cross. By Pick's formula the areas of triangles XBC and YBC are equal. Then altitudes of these triangles to side BC are equal and therefore, line XY is parallel to BC.

Remark. Consider triangle with vertices on lattice. Then the area of the triangle equals $A = i + b/2 - 1$ where i and b are the numbers of lattice points in the interior and on the boundary of the triangle respectively (Pick's formula).

6 [**8**] Let ABC be a right-angled triangle, I its incenter and B_0, A_0 points of tangency of the incircle with the legs AC and BC respectively.

Let the perpendicular dropped to AI from A_0 and the perpendicular dropped to BI from B_0 meet at point P. Prove that the lines CP and AB are perpendicular.

SOLUTION.



Let Q be the foot of perpendicular dropped from A_0 to AI. Let us denote by A_1 intersection of AI and CB, B_1 intersection of BI and CA, and R intersection of PC and AB.

Let $\angle CAB = 2\alpha$ and $\angle ABC = 2\beta$ ($2\alpha + 2\beta = 90°$). Then $\angle B_0IB_1 = \angle CBB_1 = \alpha$ and $\angle A_0IA_1 = \angle CAA_1 = \beta$. Then $\angle B_0PA_0 = \alpha + \beta = 45°$. (Indeed, in triangle PB_0A_0, $\angle B_0A_0P = 45° + \beta$, $\angle A_0B_0P = 45° + \alpha$ and therefore, $\angle B_0PA_0 = 180° - 45° - \alpha - 45° - \beta = 90° - (\alpha + \beta) = 45°$).

Consider the circle with centre C and radius B_0C ($B_0C = A_0C$). Note that $\angle B_0PA_0 = 1/2\angle B_0CA_0$. Since $\angle B_0CA_0$ is a central angle, $\angle B_0PA_0$ must be inscribed in constructed circle. It follows that triangle PCA_0 is isosceles ($PC = CA_0$) so that $\angle CPA_0 = \angle CA_0P = \beta$ and therefore $\angle RCB = 2\beta$. In triangle CRB we have: $\angle RCB = 2\beta$ and $\angle RBC = 2\alpha$. Then $\angle CRB = 2\alpha + 2\beta = 90°$.

7 [**9**] Two teams A and B play a school ping pong tournament. The team A consists of m students, and the team B consists of n students where $m \neq n$.

There is only one ping pong table to play and the tournament is organized as follows. Two students from different teams start to play while other players form a line waiting for their turn to play. After each game the first player in the line replaces the member of the same team at the table and plays with the remaining player. The replaced player then goes to the end of the line. Prove that every two players from the opposite teams will eventually play against each other.

SOLUTION. Let us separately numerate the players of each team according to their order in the original line: $a_1, a_2 \dots, a_m$ and $b_1, b_2 \dots, b_n$. The first game is played between a_1 and b_1. Observe that during the tournament the players of the same team (including a player at the table) preserve their cyclic order in a line. A player of one team can change his position relatively to a player of another team and it can be only at the table: a player can come to the table before and leave the table after a player of the opposite team.

Let us split the games into series consisting of $m + n - 2$ games each. During the first series, all first $m + n - 2$ players except a_m and b_n will play at the table. The latter two will play the first game in the second series. Similarly, a_{m-1} and b_{n-1} will be the ones who play the first game in the third series, and so on. We can see that each time, the index of each player in the first game of the series moves one position back in a cycle.

This implies that in m series every member of the team A will exit the table exactly $m-1$ times and in n series every member of the team B will exit the table exactly $n-1$ times. Therefore in mn series the members of teams A and B will exit the table exactly $n(m-1)$ and $m(n-1)$ times respectively.

Let $m > n$. Then $n(m-1) - m(n-1) = m - n \geq 1$. Consider arbitrary players a_i and b_j. After $2mn$ series a_i exits the table at least 2 times more than b_j. Therefore, there will be two consecutive exiting of a_i such that b_j remains in the line. Assuming that a_i and b_j do not meet at the table, a_i must overcome b_j in a line, which is not possible.

Remarks. 1. In fact, each pair of players of different teams meets already in the first mn series. Indeed, if m and n are relatively prime then according to the above the players meet at the beginning of a new series. Otherwise $|m-n| \geq 2$, and the proof works for mn cycles.

1^* Since a_i is exactly in the same place after m cycles and b_j is in the same place after n cycles, everything repeats after $\operatorname{lcm}(m, n)$ cycles.

2. For $n = m > 2$ the assertion fails. For example, if the players of both teams alternate, each of them plays only with his neighbours in a line.