International Mathematics

TOURNAMENT OF THE TOWNS

Senior O-Level Paper Solutions **Fall 2013**

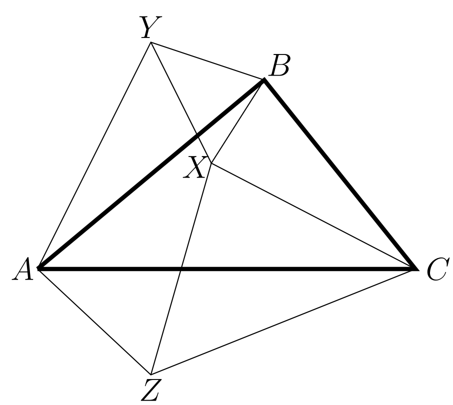
1 [**3**] Does there exist a ten-digit number such that all its digits are different and after removing any six digits we get a composite four-digit number?

ANSWER: yes.

SOLUTION. Observe that a four-digit number 1379 is divided by 7 ($1379 = 7 \times 197$) We can consider a ten-digit number in the form $1379\ldots$ where the tail is any combination of remaining digits $2, 4, 6, 8, 0, 5$. It is easy to see that this number satisfies the conditions: the remaining four digits form either 1379, either an even four-digit number, or a four-digit multiple of 5.

2 [**4**] On the sides of triangle ABC, three similar triangles are constructed with triangle YBA and triangle ZAC in the exterior and triangle XBC in the interior. (Above, the vertices of the triangles are ordered so that the similarities take vertices to corresponding vertices, for example, the similarity between triangle YBA and triangle ZAC takes Y to Z, B to A and A to C). Prove that $AYXZ$ is a parallelogram.

SOLUTION. Draw the following figure



For simplicity of notations let us denote: $\angle BAC = a$, $\angle ABC = b$, $\angle ACB = c$. Further, using similarity of triangles YBA, ZAC, and XBC let us denote

$$\angle YAB = \angle ZCA = \angle XCB = \alpha,$$
$$\angle YBA = \angle ZAC = \angle XBC = \beta$$

and

$$\angle AYB = \angle CZA = \angle CXB = \gamma.$$

Since triangle YBA is similar to triangle XBC we have $YB : AB = XB : BC$. It follows that $YB : BX = AB : BC$. Since $\angle YBA = \angle XBC$ we have $\angle YBX = b$.

Therefore triangle YBX is similar to triangle ABC. It implies that $\angle XYB = a$ while $\angle YXB = c$.

Since triangle ZAC is similar to triangle XBC, in a similar way we can show $ZC : XC = AC : BC$. Since $\angle ZCX = c$, triangle ZXC is similar to triangle ABC. Then $\angle XZC = a$ while $\angle ZXC = b$.

It implies that that:

$$\angle AYX = \angle AYB - \angle XYB = \gamma - a,$$

and

$$\angle AZX = \angle AZC - \angle XZC = \gamma - a$$

and therefore one pair of opposite angles of quadrilateral $AYXZ$ is equal. Further,

$$\angle YXZ = 360^\circ - \angle YXB - \angle ZXC - \angle BXC = 360^\circ - c - b - \gamma = \\ (180^\circ - c - b) + (180^\circ - \gamma) = a + \alpha + \beta = \angle YAZ$$

and therefore, the other pair of opposite angles of quadrilateral $AYXZ$ is equal.

Hence quadrilateral $AYXZ$ is a parallelogram.

3 [4] Denote by $[a, b]$ the least common multiple of a and b. Let n be a positive integer such that

$$[n, n+1] > [n, n+2] > \cdots > [n, n+35]. \tag{2}$$

Prove that $[n, n+35] > [n, n+36]$.

Solution. Let (n, m) denote the least common multiple of n and m. Then $(n, m) = nm/[n, m]$ and since $n(n+1) < n(n+2) < \cdots < n(n+35)$ we conclude from (2) that

$$(n, n+1) < (n, n+2) < \cdots < (n, n+35). \tag{3}$$

holds.

Then $(n, n+m) = m$ as $m = 1, \ldots, 36$,

$$[n, n+35] = n(n+35)/35 = n^2/35 + n$$

and

$$[n, n+36] = n(n+36)/36 = n^2/36 + n$$

and the latter is obviously smaller than the former.

4 [5] Eight rooks are placed on a chessboard so that no two rooks attack each other. Prove that one can always move all rooks, each by a move of a knight so that in the final position no two rooks attack each other as well. (In intermediate positions several rooks can share the same square).

Solution. Observe that condition "no two rooks attack one another" means exactly that

(a) Each horizontal has 1 rook,

(b)Each vertical has 1 rook.

Break movement into two steps:

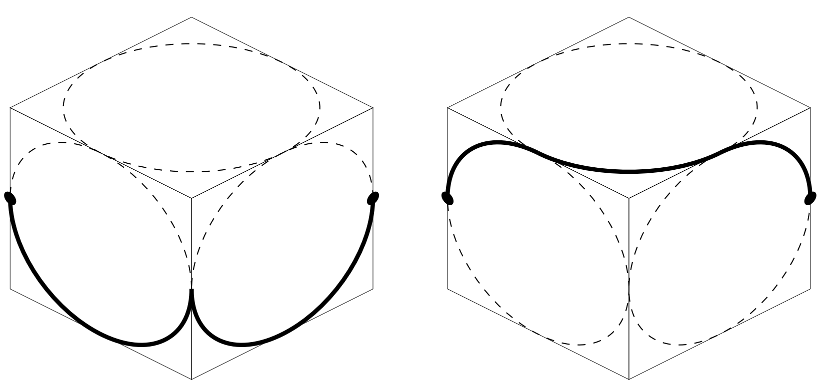
Step 1: Rooks from verticals 1,2,5,6 move 2 squares right – to verticals 3,4,7,8 respectively; rooks from verticals 3,4,7,8 move 2 squares left – to verticals 1,2,5,6 respectively. Obviously both conditions (a), (b) remains fulfilled.

Step 2: Rooks from horizontals 1,3,5,7 move 1 square up – to horizontals 2,4,7,8; rooks from horizontals 2,4,7,8 move 1 square down – to horizontals 1,3,5,7 respectively Obviously both conditions (a), (b) remains fulfilled.

As a result each rook made a knight's move.

5 [6] A spacecraft landed on an asteroid. It is known that the asteroid is either a ball or a cube. The rover started its route at the landing site and finished it at the point symmetric to the landing site with respect to the center of the asteroid. On its way, the rover transmitted its spatial coordinates to the spacecraft on the landing site so that the trajectory of the rover movement was known. Can it happen that this information is not sufficient to determine whether the asteroid is a ball or a cube?

Solution. Consider a sphere of radius r and a surface of cube with the side a with the same center. Observe that if $a = \sqrt{2}r$ the sphere touches each edge at its midpoint and therefore it intersects each face of the cube along circle of radius $r/\sqrt{2}$ in its center like on the figure below (we draw only three visible faces):



Then any path consisting of arcs of these circles belongs to both sphere and the surface of the cube and one can connect two symmetric points marked on the figure by such path. Therefore *it can happen that such information is not sufficient to determine whether the asteroid is a ball or a cube.*