1 [3] Does there exist a ten-digit number such that all its digits are different and after removing any six digits we get a composite four-digit number?

**Answer:** yes.

**Solution.** Observe that a four-digit number 1379 is divided by 7 (1379 = 7 × 197) We can consider a ten-digit number in the form 1379... where the tail is any combination of remaining digits 2, 4, 6, 8, 0, 5. It is easy to see that this number satisfies the conditions: the remaining four digits form either 1379, either an even four-digit number, or a four-digit multiple of 5.

2 [4] On the sides of triangle \( \triangle ABC \), three similar triangles are constructed with triangle \( \triangle YBA \) and triangle \( \triangle ZAC \) in the exterior and triangle \( \triangle XBC \) in the interior. (Above, the vertices of the triangles are ordered so that the similarities take vertices to corresponding vertices, for example, the similarity between triangle \( \triangle YBA \) and triangle \( \triangle ZAC \) takes \( Y \) to \( Z \), \( B \) to \( A \) and \( A \) to \( C \)). Prove that \( \triangle AYZ \) is a parallelogram.

**Solution.** Draw the following figure

![Diagram of triangle ABC with similar triangles YBA, ZAC, and XBC]

For simplicity of notations let us denote: \( \angle BAC = a \), \( \angle ABC = b \), \( \angle ACB = c \). Further, using similarity of triangles \( \triangle YBA \), \( \triangle ZAC \), and \( \triangle XBC \) let us denote

\[
\angle YAB = \angle ZCA = \angle XCB = \alpha,
\]

\[
\angle YBA = \angle ZAC = \angle XBC = \beta
\]

and

\[
\angle AYB = \angle CZA = \angle CXB = \gamma.
\]

Since triangle \( \triangle YBA \) is similar to triangle \( \triangle XBC \) we have \( YB : AB = XB : BC \). It follows that \( YB : BX = AB : BC \). Since \( \angle YBA = \angle XBC \) we have \( \angle YBX = b \).
Therefore triangle $YBX$ is similar to triangle $ABC$. It implies that $\angle XYB = a$ while $\angle YXB = c$.

Since triangle $ZAC$ is similar to triangle $XBC$, in a similar way we can show $ZC : XC = AC : BC$. Since $\angle ZCX = c$, triangle $ZXC$ is similar to triangle $ABC$. Then $\angle XZC = a$ while $\angle ZXC = b$.

It implies that that:

$$\angle AYX = \angle AYB - \angle XYB = \gamma - a,$$
and

$$\angle AZX = \angle AZC - \angle XZC = \gamma - a$$
and therefore one pair of opposite angles of quadrilateral $AYXZ$ is equal. Further,

$$\angle YXZ = 360^\circ - \angle YXB - \angle ZXC - \angle BXC = 360^\circ - c - b - \gamma = (180^\circ - c - b) + (180^\circ - \gamma) = a + \alpha + \beta = \angle YAZ$$
and therefore, the other pair of opposite angles of quadrilateral $AYXZ$ is equal.

Hence quadrilateral $AYXZ$ is a parallelogram.

3 [4] Denote by $[a, b]$ the least common multiple of $a$ and $b$. Let $n$ be a positive integer such that

$$[n, n + 1] > [n, n + 2] > \cdots > [n, n + 35]. \quad (2)$$

Prove that $[n, n + 35] > [n, n + 36]$.

Solution. Let $(n, m)$ denote the least common multiple of $n$ and $m$. Then $(n, m) = nm/[n, m]$ and since $n(n + 1) < n(n + 2) < \cdots < n(n + 35)$ we conclude from (2) that

$$(n, n + 1) < (n, n + 2) < \cdots < (n, n + 35). \quad (3)$$
holds.

Then $(n, n + m) = m$ as $m = 1, \ldots, 36$,

$$[n, n + 35] = n(n + 35)/35 = n^2/35 + n$$
and

$$[n, n + 36] = n(n + 36)/36 = n^2/36 + n$$
and the latter is obviously smaller than the former.

4 [5] Eight rooks are placed on a chessboard so that no two rooks attack each other. Prove that one can always move all rooks, each by a move of a knight so that in the final position no two rooks attack each other as well. (In intermediate positions several rooks can share the same square).
Solution. Observe that condition “no two rooks attack one another” means exactly that

(a) Each horizontal has 1 rook,
(b) Each vertical has 1 rook.

Break movement into two steps:

Step 1: Rooks from verticals 1, 2, 5, 6 move 2 squares right – to verticals 3, 4, 7, 8 respectively; rooks from verticals 3, 4, 7, 8 move 2 squares left – to verticals 1, 2, 5, 6 respectively. Obviously both conditions (a), (b) remains fulfilled.

Step 2: Rooks from horizontals 1, 3, 5, 7 move 1 square up – to horizontals 2, 4, 7, 8; rooks from horizontals 2, 4, 7, 8 move 1 square down – to horizontals 1, 3, 5, 7 respectively. Obviously both conditions (a), (b) remains fulfilled.

As a result each rook made a knight’s move.

5 [6] A spacecraft landed on an asteroid. It is known that the asteroid is either a ball or a cube. The rover started its route at the landing site and finished it at the point symmetric to the landing site with respect to the center of the asteroid. On its way, the rover transmitted its spatial coordinates to the spacecraft on the landing site so that the trajectory of the rover movement was known. Can it happen that this information is not sufficient to determine whether the asteroid is a ball or a cube?

Solution. Consider a sphere of radius \( r \) and a surface of cube with the side \( a \) with the same center. Observe that if \( a = \sqrt{2}r \) the sphere touches each edge at its midpoint and therefore it intersects each face of the cube along circle of radius \( r/\sqrt{2} \) in its center like on the figure below (we draw only three visible faces):

Then any path consisting of arcs of these circles belongs to both sphere and the surface of the cube and one can connect two symmetric points marked on the figure by such path. Therefore it can happen that such information is not sufficient to determine whether the asteroid is a ball or a cube.