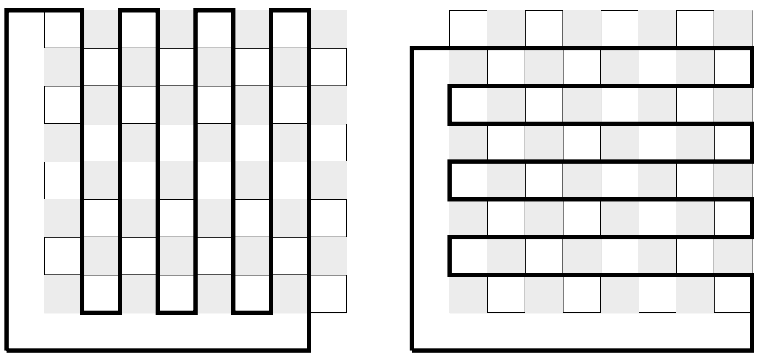
International Mathematics

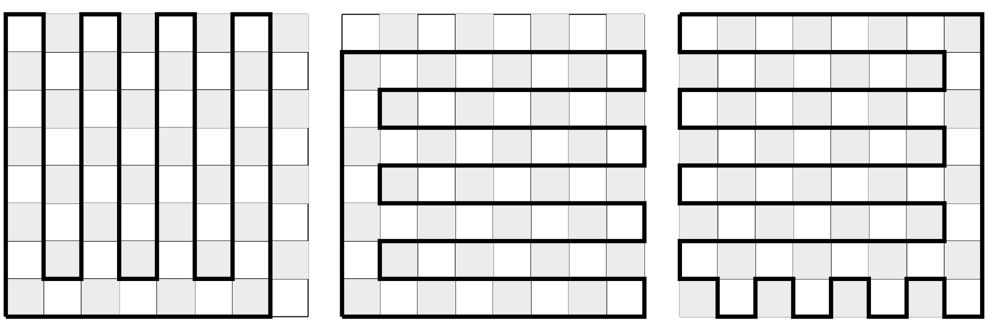
TOURNAMENT OF THE TOWNS

Senior A-Level Solutions **Fall 2013**

1. Pete drew a square in the plane, divided it into 64 equal square cells and painted it in a chess board fashion. He chose some cell and an interior point in it. Basil can draw any polygon (without self-intersections) in the plane and ask Pete whether the chosen point is inside or outside this polygon. What is the minimal number of questions sufficient to determine whether the chosen point is black or white?



SOLUTION 1. One question is not enough because a polygon containing all white points and no black point has to be self-intersecting. However two questions are enough: if a point belongs to just one polygon then it is white, and if a point belongs to both or none then it is black.



(a) Polygon 1 (b) Polygon 2b (c) Polygin 2c

SOLUTION 2 [Nikita Kapustin, gr. 11, Richmond Hill H.S.]. If the point is outside of the Polygon 1 then it is confined to verticals 2,4,6,8 and we determine the colour by drawing Polygon 2b.

If the point is inside the Polygon 1 then it is confined either to verticals 1,3,5,7 or to horizontal 1 and we determine the colour drawing Polygon 2c: it contains horizontals 2,4,6,8 and only white squares from horizontal 1. Therefore if the point is inside Polygon 2c it is in a white square; if the point is outside Polygon 2c it is in a black square.

2. Find all positive integers n for which the following statement holds:

For any two polynomials $P(x)$ and $Q(x)$ of degree n there exist monomials ax^k and bx^ℓ, $0 \leq k, \ell \leq n$, such that the graphs of $P(x)+ax^k$ and $Q(x)+bx^\ell$ have no common points.

Answer: All even integers n and $n = 1$.

Solution. The statement can be reformulated as: "for any polynomial $R(x)$ with $\deg R \leq n$ there exist monomials ax^k and bx^ℓ with $0 \leq k < \ell \leq n$ such that the $R(x) + ax^k + bx^\ell$ has no 0."

Let $n \geq 3$ be odd. Then for $R(x) = x^n + x$ we must pick up $bx^\ell = -x^n$ (otherwise we have a polynomial of odd degree which has a 0). We have $R(x) - x^n = x$ and if $k \geq 1$ then $R(x) + bx^\ell + ax^k$ is 0 as $x = 0$; if $k = 0$ we have $x + a$ which has 0 as well.

For $n = 1$ we can always make $R(x) + bx + a \equiv 1$ which does not vanish.

Let n be even. Then add a monomial of degree n to make the leading coefficient positive. The resulting polynomial is bounded from below by some M. Adding the constant $1 - M$, we make the minimal value equal to 1 and get the required polynomial.

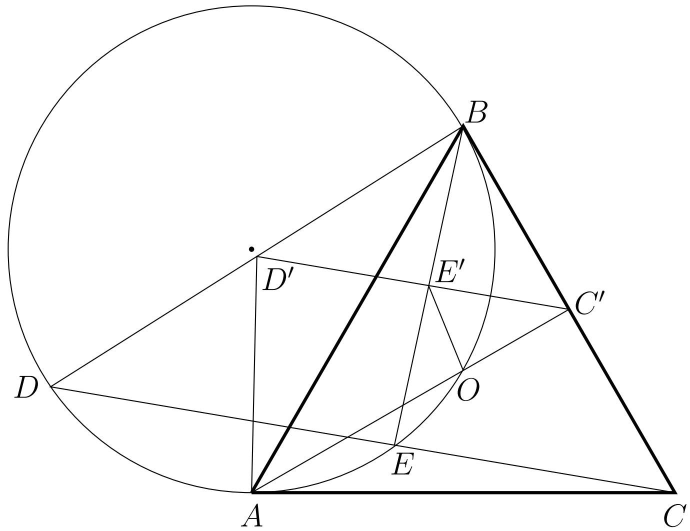
3. Let ABC be an equilateral triangle with centre O. A line through C meets the circumcircle of triangle AOB at points D and E. Prove that points A, O and the midpoints of segments BD, BE are concyclic.

Solution. Let C' be the midpoint of BC, and E be between C and D. Let E' be the midpoint of BE; then E' belongs to midline $C'D'$ of the triangle CBD.

In the equilateral triangle ABC vertex A, center O and point C' are collinear. The angle ABC is equal to the half of arc AOB and therefore BC is a tangent line of the circumcircle of triangle AOB.

Then

$$C'E' \cdot C'D' = \frac{1}{4} CE \cdot CD = \frac{1}{4} CB^2 = C'B^2 = C'O \cdot C'A.$$



By Intersecting Chord Theorem if quadrilateral $AOE'D'$ was cyclic then this equality would hold. However converse statement is also true: since this equality holds, this quadrilateral is cyclic.

4. Is it true that every integer is a sum of finite number of cubes of distinct integers?

SOLUTION. Observe that

$$(n+7)^3-(n+6)^3-(n+5)^3+(n+4)^3-(n+3)^3+(n+2)^3+(n+1)^3-n^3=48.$$

On the other hand, $(48k+1)^3$ for any k is comparable with 1 modulo 48. Summing up such cubes we obtain sums with all possible residues on division by 48 and then, adding or subtracting a suitable number of combinations equal to 48 and formed of distinct integers we get any integer with the same residue.

5. Do there exist two integer-valued functions f and g such that for every integer x we have

(a) $f(f(x))=x, \quad g(g(x))=x, \quad f(g(x))>x, \quad g(f(x))>x$?

(b) $f(f(x))<x, \quad g(g(x))<x, \quad f(g(x))>x, \quad g(f(x))>x$?

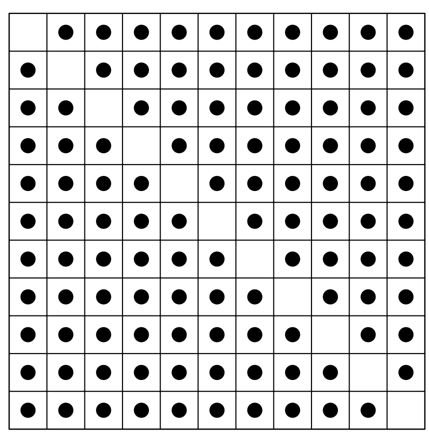
SOLUTION: (a) Such functions do not exist. Indeed, $f(f(x))=x \implies g(f(f(x))=g(x)$, but $g(f(f(x)))>f(x)$ so $g(x)>f(x)$. Similarly $f(x)>g(x)$. A contradiction.

(b) Solution. Such functions exist. For instance, call even integers *associated with function f* and odd ones *associated with function g*.

Let these functions map any associated x to $-|x| - 2$ and any other x to $|x| + 1$. Observe that all values of each function are associated with it. Clearly $|f(x)| > |x|$, hence $f(f(x)) = -|f(x)| < -|x| - 2 < x$ and $g(f(x)) = |f(x)| + 1 > |x| + 1 > x$. The remaining two inequalities are checked in the same way.

6. On a table, there are 11 piles of ten stones each. Pete and Basil play the following game. In turns they take 1, 2 or 3 stones at a time: Pete takes stones from any single pile while Basil takes stones from different piles but no more than one from each. Pete moves first. The player who cannot move, loses. Which of the players, Pete or Basil, can guarantee a victory regardless of the opponent's play?

Solution. Let us position stones as on the picture so that piles correspond to columns. Peter must take several stones from one column and Basil must take them from different columns. Basil' strategy is to make moves symmetric to those of Peter with respect to empty diagonal. Since a row symmetric to a column has no common stones with it, Basil can each time restore the broken symmetry, so he always has a move. Since the number of stones is finite, eventually Peter loses.



7. A closed broken self-intersecting line is drawn in the plane. Each of the links of this line is intersected exactly once and no three links intersect at the same point. Further, here are no self-intersections at the vertices and no two links have a common segment. Can it happen that every point of self-intersection divides both links in halves?

Answer: no.

Solution. First, we need a following

Theorem. *A closed piecewise smooth line with simple self-intersections (no tangency between links – which automatically holds for broken lines; no multiple intersections in the same point, no intersections in the vertices) divides a plane into several regions which can "chess like" painted so that the parts of the same colour have no common segments of the boundary.*

Proof. We can assume that the line contains no vertical segments (otherwise we achieve this by turning by some angle). Consider a point M not on the line and let us consider a vertical ray from M upward. We call the region R *even* if for point M in R such ray intersects the line even number of times and *odd* otherwise.

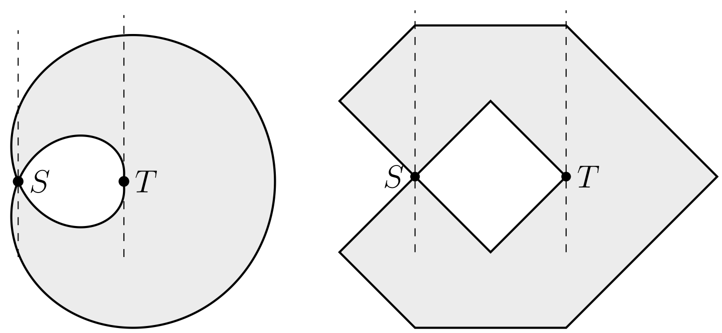


Figure 4: S is a point of self-intersection, T is a point of tangency.

To justify this definition we need to prove that it does not depend on the position of M in R, but it is true: indeed, if we move inside R up along the ray, the number of intersections does not change. If we move M inside R horizontally then this number can change when the ray passes through a point of self-intersection or tangency; it can pass simultaneously through several such points, but each of such changes brings either 0 or ± 2 so that evenness does not change.

Now if we move up or down into adjacent region this number changes by ± 1. Therefore we can colour each even region into white and each odd region into black. □

Now, after this theorem is proven let us consider the broken line which divides plane into several regions which are coloured according to the theorem and assume that an unbounded region is white.

Let us consider segments of the broken line (half-links) such that if two links AB and CD intersect in S then we consider separately AS, BS, CS, DS. Let us orient such links so that each black region R is counterclockwise oriented. Observe that AS and BS should have opposite orientations (*):

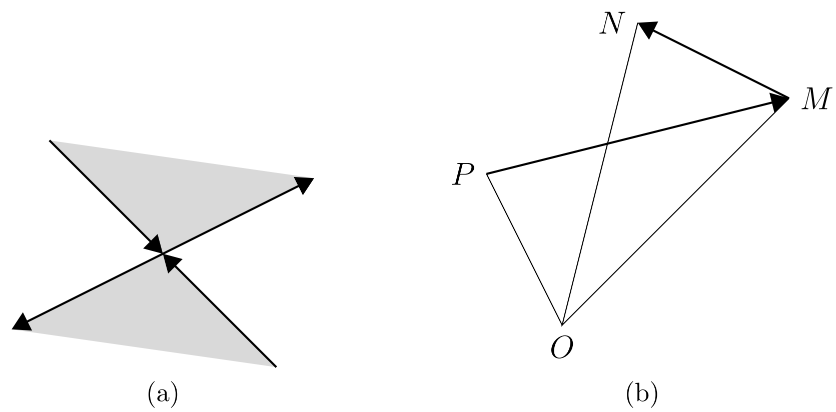


Figure 5: Orientation (a) and Oriented area (b): Oriented area of OMN is positive and of OPM is negative.

Let us select point O and for all segments MN let us consider the *oriented area* of OMN (orientation is due to orientation of MN – the area is positive if the vector MN is seen from O counterclockwise and negative otherwise).

Let us sum up oriented areas over all segments. Calculating sums in black regions, for each black region we get its area (since it is counterclockwise oriented). Indeed, if a black region is a triangle, it is obvious:

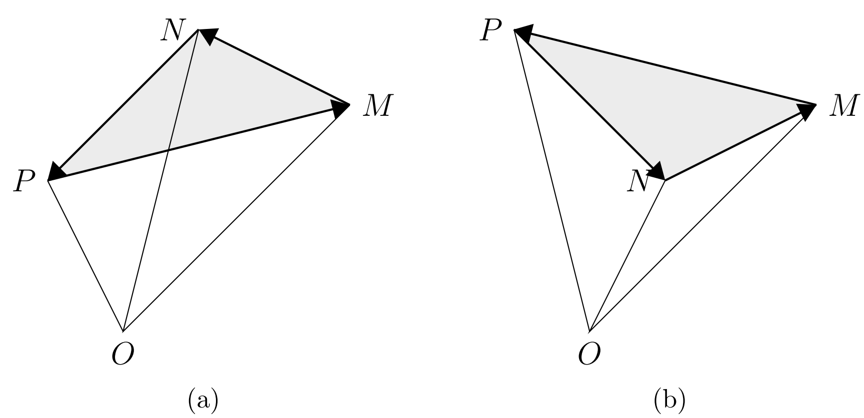


Figure 6: Sum of oriented areas of OMN, ONP and OPM equals area of MNP.

If black region is a n-gon, we prove it by induction. Assume that for $m < n$ it is true. We can cut n-gon $M_1 M_2 \dots M_n$ by some diagonal, say, $M_1 M_m$ into m-gon $M_1 \dots M_m$ and $n - m + 2$-gon $M_1 M_m \dots M_n$; both of them are counterclockwise oriented and each area equals to the sum of oriented areas of triangles. Adding we get the sum of oriented areas of troangles $O\overrightarrow{M_k M_{k+1}}$ with $k = 1, \dots n$ ($M_{n+1} = M_1$) plus oriented areas of $O\overrightarrow{M_1 M_m}$ and $O\overleftarrow{M_1 M_m}$ which cancel one another.

So the total sum equals the total area of black regions and is positive.

On the other hand, each vector enters in the total sum together with the opposite one in virtue of (*) and therefore total sum must be 0. A contradiction.