International Mathematics

TOURNAMENT OF THE TOWNS

Junior O-Level Paper Solutions **Fall 2013**

1 [**3**] In a wrestling tournament, there are 100 participants, all of different strengths. The stronger wrestler always wins over the weaker opponent. Each wrestler fights twice and those who win both of their fights are given awards. What is the least possible number of awardees?

Answer: 1.

Solution. Arrange participants by strength a_1 (the weakest), $a_2, a_3, \dots a_{100}$ (the strongest).

Obviously, a_{100} is one of the winners. Let wrestlers in the first round be paired as follows: $a_{100} - a_{99}$, $a_{98} - a_{97}$,..., $a_2 - a_1$, then $a_1, a_3, a_5, \dots a_{99}$ are losers.

Let the second round be paired as follows: $a_{100} - a_1$, $a_{99} - a_{98}$,..., $a_3 - a_2$, then $a_2, a_4, a_6, \dots, a_{98}$ are losers. Therefore the only participant who won in both rounds is a_{100}.

2 [**4**] Does there exist a ten-digit number such that all its digits are different and after removing any six digits we get a composite four-digit number?

Answer: yes.

Solution. Observe that a four-digit number 1379 is divided by 7 ($1379 = 7 \times 197$) We can consider a ten-digit number in the form $1379\dots$ where the tail is any combination of remaining digits $2, 4, 6, 8, 0, 5$. It is easy to see that this number satisfies the conditions: the remaining four digits form either 1379, either an even four-digit number, or a four-digit multiple of 5.

3 [**4**] Denote by (a, b) the greatest common divisor of a and b. Let n be a positive integer such that

$$(n, n + 1) < (n, n + 2) < \dots < (n, n + 35). \qquad (1)$$

Prove that $(n, n + 35) < (n, n + 36)$.

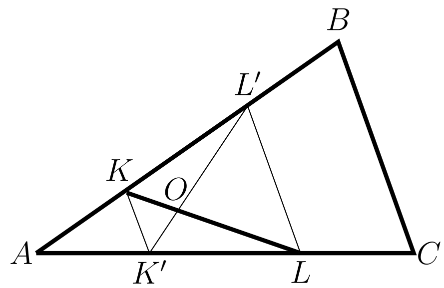
Solution. First we need

Lemma. $(n, n + m) \leq m$.

Proof. Indeed, if p divides both n and $(n + m)$ it also divides their difference which is m. □

Since $(n, n + 1) = 1$ and $(n, n + k)$ increases for $k = 1, \dots, 35$ then this lemma implies that for all $m = 1, \dots, 35$ we have $(n, n+m) = m$ and therefore n is divisible by m. In particular n is divisible by both 4 and 9 and therefore it is divisible by 36. Then $n + 36$ is also divisible by 36 and $(n, n + 36) = 36 > (n, n + 35) = 35$.

4 [**5**] Let ABC be an isosceles triangle. Points K and L are chosen on lateral sides AB and AC respectively so that $AK = CL$ and $\angle ALK + \angle LKB = 60°$. Prove that $KL = BC$.



Solution. Let us mark points K' and L' on sides AC and AB respectively so that $AK' = AK$ and $L'B = LC$. Since triangle ABC is isosceles, the lines KK', LL' and BC are parallel. Let $AB = AC = b$, $AK = LC = a$, $BC = c$, $KK' = s$ and $LL' = t$.

Note that triangles AKK', ALL' and ABC are similar. Therefore we have $s/c = a/b$ and $t/c = (b - a)/b$.

Because of symmetry, triangles KOK' and OLL' are isosceles ($LO = LO'$ and $KO = K'O$) and since $60° = \angle ALK + \angle LKB = \angle ALK + \angle L'K'C$ we have $\angle LOL' = KOK' = 60°$. This implies that triangles KOK' and OLL' are equilateral. Finally, $KL = KO + OL = s + t = c(a/b + (b - a)/b) = c = BC$.

5 [**6**] Eight rooks are placed on a chessboard so that no two rooks attack each other. Prove that one can always move all rooks, each by a move of a knight so that in the final position no two rooks attack each other as well. (In intermediate positions several rooks can share the same square).

Solution. Observe that condition "no two rooks attack one another" means exactly that

(a) Each horizontal has 1 rook,

(b) Each vertical has 1 rook.

We break movement into two steps:

Step 1: Rooks from verticals 1,2,5,6 move 2 squares right – to verticals 3,4,7,8 respectively; rooks from verticals 3,4,7,8 move 2 squares left – to verticals 1,2,5,6 respectively. Obviously both conditions (a), (b) remains fulfilled.

Step 2: Rooks from horizontals 1,3,5,7 move 1 square up – to horizontals 2,4,7,8; rooks from horizontals 2,4,7,8 move 1 square down – to horizontals 1,3,5,7 respectively Obviously both conditions (a), (b) remains fulfilled.

As a result each rook made a knight's move.