International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Solutions **Fall 2013**

1. There are 100 red, 100 yellow and 100 green sticks. One can construct a triangle using any three sticks all of different colour. Prove that there is a colour such that one can construct a triangle using any three sticks of this colour.

Solution. For each of three colours (a, b,and c) consider two smallest sticks and the largest stick and denote these (x_1, x_2, x) respectively. Assume that there is no such colour that one can always construct a triangle using any three sticks of this colour. This implies: $x_1 + x_2 \le x$. Without loss of generality assume that $a_1 \le b_1 \le c_1$. Then $a_1 + b_1 \le c_1 + c_2 \le c$.

Contradiction: one cannot construct a triangle using any three sticks all of different colours.

2. A math teacher chose 10 consequent positive integers and submitted them to Pete and Basil. Each boy should split these numbers in pairs and calculate the sum of products of numbers in pairs. Prove that the boys can pair the numbers differently so that the resulting sums are equal.

Solution. Let consecutive numbers be in the form $n+1, n+2, n+3, \dots, n+10$. One can check that $P_1 = P_2$, where

$$P_1 = (n+1)(n+8) + (n+2)(n+7) + (n+3)(n+6) + (n+4)(n+5) + (n+9)(n+10)$$

and

$$P_2 = (n+1)(n+10) + (n+2)(n+3) + (n+4)(n+5) + (n+6)(n+7) + (n+8)(n+9).$$

3. Assume that C is a right angle of triangle ABC and N is a midpoint of the semicircle, constructed on CB as on diameter externally. Prove that AN divides the bisector of angle C in halves.

Solution. Extend segment BN to intersect line AC at some point K. In triangle BCK the altitude CN is also a bisector, thus $KN = NB$. Angles BCL and CBK are equal to $45°$, hence the bisector CL is parallel to BK. Therefore in triangle ABK the median AN bisects CL as well.

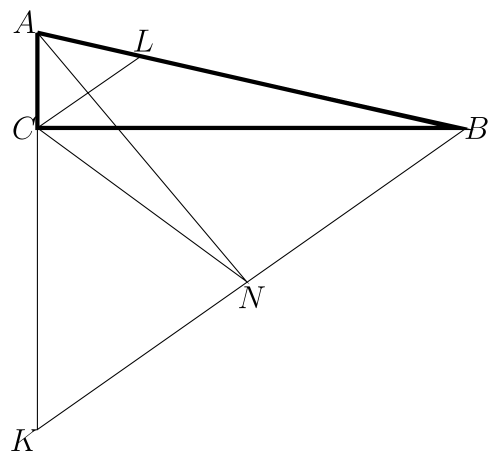
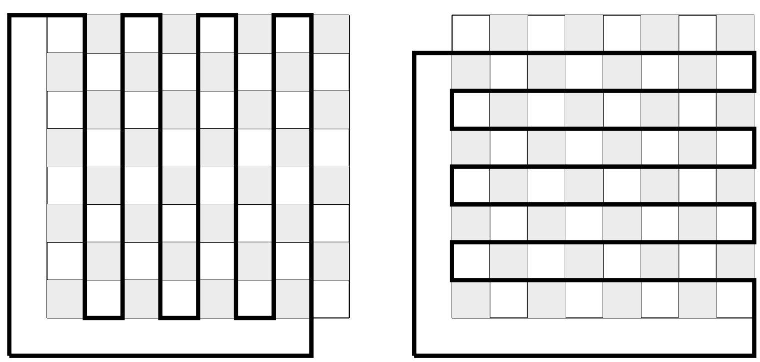


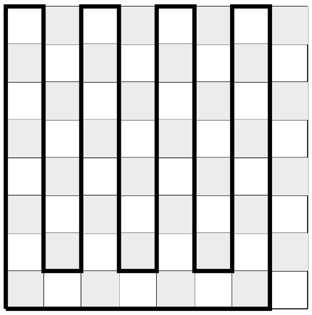
Figure 1: To Problem 3

4. Pete drew a square in the plane, divided it into 64 equal square cells and painted it in a chess board fashion. He chose some cell and an interior point in it. Basil can draw any polygon (without self-intersections) in the plane and ask Pete whether the chosen point is inside or outside this polygon. What is the minimal number of questions sufficient to determine whether the chosen point is black or white?

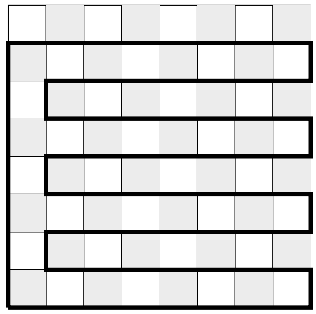


Solution 1. One question is not enough because a polygon containing all white points and no black point has to be self-intersecting. However two questions are enough: if a point belongs to just one polygon then it is white, and if a point belongs to both or none then it is black.

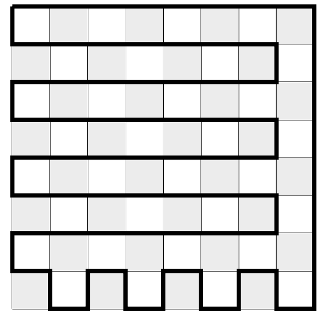
Solution 2 [Nikita Kapustin, gr. 11, Richmond Hill H.S.]. If the point is outside of the Polygon 1 then it is confined to verticals 2,4,6,8 and we determine the colour by drawing Polygon 2b.



(a) Polygon 1



(b) Polygon 2b



(c) Polygin 2c

Figure 2: To problem 4

If the point is inside of the Polygon 1 then it is confined either to verticals 1,3,5,7 or to horizontal 1 and we determine the colour drawing Polygon 2c: it contains horizontals 2,4,6,8 and only white squares from horizontal 1. Therefore if the point is inside of Polygon 2c it is in a white square; if the point is outside of Polygon 2c it is in a black square.

5. A 101-gon is inscribed in a circle. From each vertex of this polygon a perpendicular is dropped to the opposite side or its extension. Prove that at least one perpendicular drops to the side.

Solution. We call 50 consecutive arcs a *train*. Train is *long* if its total measure is at least 180°, otherwise it is *short*.

If none of the perpendiculars from vertices to the opposite sides lands inside of the edge (vertices excluded!), then

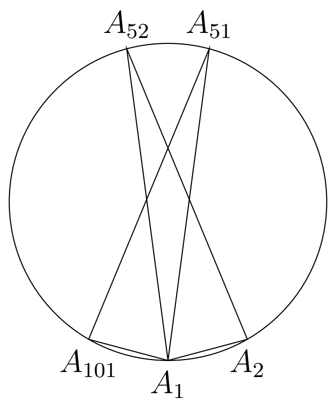
(*) Out of two disjoint (not having common arcs) trains one must be long (and another short).

Indeed, if two disjoint trains ($\overset{\frown}{AB}$ and $\overset{\frown}{AC}$ are short then the in the triangle ABC angles $\angle B$ and $\angle C$ are acute and a perpendicular dropped from A to the opposite edge BC land inside BC.

However for each train there are two trains disjoint from it: for a long train there are two short trains and for a short train due to (*) there are two long trains. Then moving long train in any direction by one arc we have a long train again. Then all trains must be long. Contradiction.

Solution 2 [Jennifer Guo, gr. 10, Marc Garneau C.I.].

Assume that the perpendicular from any vertex of polygon drops to the extension of the opposite side. Then in every *main triangle* (i.e. trianle



formed by some side and two diagonals connecting its ends to the opposite vertex) one of base angles must be either obtuse or right.

Consider triangle $A_{101}A_{51}A_1$ and assume that $\angle A_{51}A101A_1 \geq 90^\circ$; then supporting arc $(A_{51}A_2A_1) \geq 180^\circ$. Consider adjacent triangle $A_1A_{52}A_2$ and observe that $\angle A_{52}A_2A_1 < 90^\circ$. Indeed, otherwise supporting arc $(A_{52}A_{101}A_1) \geq 180^\circ$. However together arcs $(A_{51}A_2A_1)$ and $(A_{52}A_{101}A_1)$ are less than 360°.

Therefore in virtue of our assumption $\angle A_{52}A_1A_2 \geq 90^\circ$. Repeating these arguments we conclude that $\angle A_{53}A_2A_3 \geq 90^\circ$ and so on. In other words, each counter-clockwise arc $(A_1A_{51}), (A_2A_{52}), (A_3A_{53}), \ldots, (A_{51}A_{101})$ is at least 180°. But we know that $(A_{51}A_{101})$ is less than 180°. Contradiction.

6. The number

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{2n-1} - \frac{1}{2n}$$

is represented as an irreducible fraction. If $3n+1$ is a prime number, prove that the numerator of this fraction is a multiple of $3n+1$.

Solution [Jennifer Guo, gr. 10, Marc Garneau C.I.]. Observe that if $3n+1$ is prime then n is even (otherwise $3n+1$ is even and greater than 2).

Let us rewrite the sum (denote it by Σ) as

$$\Sigma = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2n-1} + \frac{1}{2n} - 2\Big(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2n}\Big)$$
$$= \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}.$$

Here we have n terms and since n is even we can pair them

$$
\begin{aligned}
&\Big(\frac{1}{n+1}+\frac{1}{2n}\Big)+\Big(\frac{1}{n+2}+\frac{1}{2n-1}\Big)+\ldots+\Big(\frac{1}{\frac{3}{2}n}+\frac{1}{\frac{3}{2}n+1}\Big)\\
&\quad=\frac{3n+1}{(n+1)(2n)}+\frac{3n+1}{(n+2)(2n-1)}+\ldots+\frac{3n+1}{(\frac{3}{2}n)(\frac{3}{2}n+1)}=\frac{(3n+1)p}{q}
\end{aligned}
$$

where p/q is irreducible and equal to $\frac{1}{(n+1)(2n)}+\frac{1}{(n+2)(2n-1)}+\ldots+\frac{1}{(\frac{3}{2}n)(\frac{3}{2}n+1)}$. Obviously q and $(3n+1)$ are coprime as $(3n+1)$ is prime and all prime factors of q do not exceed $2n < 3n+1$. Therefore $(3n+1)p/q$ is also an irreducible fraction with the numerator divisible by $(3n+1)$.

7. On a table, there are 11 piles of ten stones each. Pete and Basil play the following game. In turns they take 1, 2 or 3 stones at a time: Pete takes stones from any single pile while Basil takes stones from different piles but no more than one from each. Pete moves first. The player who cannot move, loses. Which of the players, Pete or Basil, can guarantee a victory regardless of the opponent's play?

Solution. Let us position stones as on the picture so that piles correspond to columns. Peter must take several stones from one column and Basil must take them from different columns. Basil' strategy is to make moves symmetric to those of Peter with respect to empty diagonal. Since a row symmetric to a column has no common stones with it, Basil can each time restore the broken symmetry, so he always has a move. Since the number of stones is finite, eventually Peter loses.

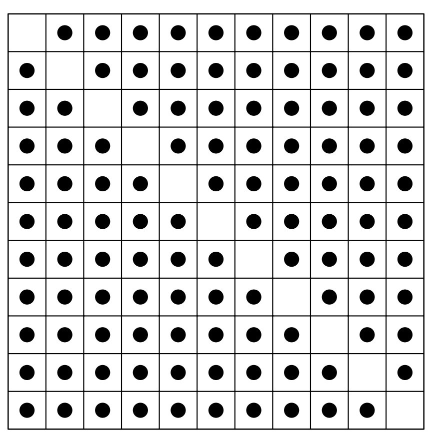


Figure 3