International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Spring 2012[1].**

1. Each vertex of a convex polyhedron lies on exactly three edges, at least two of which have the same length. Prove that the polyhedron has three edges of the same length.

2. The cells of a $1 \times 2n$ board are labelled $1, 2, \ldots, n, \ -n, \ \ldots, \ -2, \ -1$ from left to right. A marker is placed on an arbitrary cell. If the label of the cell is positive, the marker moves to the right a number of cells equal to the value of the label. If the label is negative, the marker moves to the left a number of cells equal to the absolute value of the label. Prove that if the marker can always visit all cells of the board, then $2n + 1$ is prime.

3. Consider the points of intersection of the graphs $y = \cos x$ and $x = 100\cos(100y)$ for which both coordinates are positive. Let a be the sum of their x-coordinates and b be the sum of their y-coordinates. Determine the value of $\frac{a}{b}$.

4. A quadrilateral $ABCD$ with no parallel sides is inscribed in a circle. Two circles, one passing through A and B, and the other through C and D, are tangent to each other at X. Prove that the locus of X is a circle.

5. In an 8×8 chessboard, the rows are numbers from 1 to 8 and the columns are labelled from a to h. In a two-player game on this chessboard, the first player has a White Rook which starts on the square b2, and the second player has a Black Rook which starts on the square c4. The two players take turns moving their rooks. In each move, a rook lands on another square in the same row or the same column as its starting square. However, that square cannot be under attack by the other rook, and cannot have been landed on before by either rook. The player without a move loses the game. Which player has a winning strategy?

Note: The problems are worth 4, 4, 5, 5 and 5 points respectively.

[1]Courtesy of Professor Andy Liu.

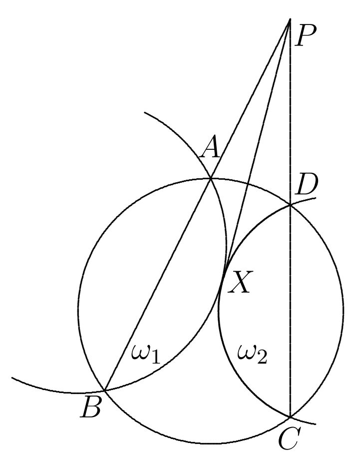
Solution to Senior O-Level Spring 2012

1. Let the number of vertices be v and the number of edges be e. Since each vertex lies on exactly 3 edges and each edge joins exactly 2 vertices, $3v = 2e$. Suppose to the contrary that the polyhedron does not have three edges of equal length. Then each vertex has a pair of equal edges of length different from any other edge. Hence the total number of edges is at least $2v$, but $3v = 2e \geq 4v$ is a contradiction.

2. Suppose $2n + 1$ is not prime. Then it has a prime divisor $p < 2n + 1$. If the marker starts on a cell whose label is divisible by p, it must move in either direction by a number of spaces equal to a multiple of p. We claim that the cells whose labels are divisible by p are evenly spaced. Then the marker must stay on cells whose labels are divisible by p, and cannot visit all cells. The claim certainly holds among the cells with positive labels and among those with negative labels. To see that it also holds across the two sides, simply add $2n + 1$ to each of the negative labels. Then a label is divisible by p after the modification if and only if it is divisible by p before the modification. The desired result follows since the modified labels are 1, 2, $\ldots$, $2n$ from left to right.

3. **Solution by Olga Ivrii.**
Define $X = \frac{x}{10}$ and $Y = 10y$. Then the graphs become the symmetric pair $Y = 10\cos(10X)$ and $X = 10\cos(10Y)$. Now X and Y are both positive if and only if both x and y are positive. Let A be the sum of the X-coordinates and B be the sum of the Y-coordinates.for which both X and Y are positive. Then $A = \frac{a}{10}$ and $B = 10b$ by definition, and $A = B$ by symmetry. Hence $\frac{a}{b} = \frac{10A}{\frac{B}{10}} = 100$.

4. **First Solution by Central Jury:**
Draw the circumcircle of $ABCD$, a circle ω_1 through A and B and a circle ω_2 through C and D such that ω_1 and ω_2 are tangent to each other. Let the lines AB and CD omtersect at P, which is necessarily outside all three circle. Let the line PX intersect ω_1 again at Y_1 and ω_2 again at Y_2. We have $PA \cdot PB = PC \cdot PD = k$, $PA \cdot PB = PX \cdot PY_1$ and $PC \cdot PD = PX \cdot PY_2$. It follows that Y_1 and Y_2 coincide. Since this point lies on both ω_1 and ω_2, it must also coincide with X. Hence PX^2 is equal to the constant k, so that the locus of X is a circle with centre P.



Second Solution:

Perform an inversion with respect to A, and let B', C', D' and X' be the respective images of B, C, D and X. The image of the fixed circle is the line passing through B', C' and D', with B' not between C' and D'. The image of the circle passing through C and D is a circle ω passing through C' and D', and the image of the circle passing through A and B is a line passing through B' and tangent to ω at X'. Hence $(B'X')^2 = B'C' \cdot B'D'$. Since B', C' and D' are fixed points, X' is at a constant distance from B', so that its locus is a circle λ with centre B'. It follows that the locus of X is the pre-image of λ, which is either a straight line or a circle. In order for it to be a straight line, λ must pass through A, so that $(AB')^2 = B'C' \cdot B'D'$. Let r be the radius of inversion. Then $(\frac{r}{AB})^2 = \frac{r^2 BC}{AB \cdot AC} \cdot \frac{r^2 BD}{AB \cdot AD}$, which simplifies to $\frac{AC}{BC} = \frac{BD}{AD}$. Since $\angle ACB = \angle BDA$, triangles ACB and BDA are similar, so that $\angle ABC = \angle BAD$. It follows that CD is parallel to AB, but we are given that $ABCD$ has no parallel sides. Hence the locus of X is indeed a circle.

5. The second player has a winning strategy. Divide the eight rows into four pairs (1,3), (2,4), (5,7) and (6,8), and the eight columns also into four pairs (b, c), (d, e), (f, g) and (h, a). Then divide the sixty-four squares into thirty-two pairs. Two squares are in the same pair if and only if they are on two different rows which form a pair, and on two different columns which also form a pair. Thus the starting squares of the two rooks form a pair. The second player's strategy is to move the Black Rook to the square which forms a pair with the square where the White Rook has just landed. First, this can always be done, because if the White Rook stays on its current row, the Black Rook will do the same, and if the White Rook stays on its current column, the Black Rook will do the same. Second, since the square on which the White Rook has just landed cannot have been landed on before, the square to which the Black Rook is moving has never been landed on before, since the squares are occupied by the two rooks in pairs. Third, the Black Rook will not be under attack by the White Rook since the two squares in the same pair are on different rows and on different columns. Hence the second player always has a move, and can simply wait for the first player to run out of moves.