International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Spring 2012[1].**

1. Each vertex of a convex polyhedron lies on exactly three edges, at least two of which have the same length. Prove that the polyhedron has three edges of the same length.

2. The cells of a $1 \times 2n$ board are labelled $1, 2, \ldots, n, -n, \ldots, -2, -1$ from left to right. A marker is placed on an arbitrary cell. If the label of the cell is positive, the marker moves to the right a number of cells equal to the value of the label. If the label is negative, the marker moves to the left a number of cells equal to the absolute value of the label. Prove that if the marker can always visit all cells of the board, then $2n+1$ is prime.

3. Consider the points of intersection of the graphs $y = \cos x$ and $x = 100\cos(100y)$ for which both coordinates are positive. Let a be the sum of their x-coordinates and b be the sum of their y-coordinates. Determine the value of $\frac{a}{b}$.

4. A quadrilateral $ABCD$ with no parallel sides is inscribed in a circle. Two circles, one passing through A and B, and the other through C and D, are tangent to each other at X. Prove that the locus of X is a circle.

5. In an 8×8 chessboard, the rows are numbers from 1 to 8 and the columns are labelled from a to h. In a two-player game on this chessboard, the first player has a White Rook which starts on the square b2, and the second player has a Black Rook which starts on the square c4. The two players take turns moving their rooks. In each move, a rook lands on another square in the same row or the same column as its starting square. However, that square cannot be under attack by the other rook, and cannot have been landed on before by either rook. The player without a move loses the game. Which player has a winning strategy?

Note: The problems are worth 4, 4, 5, 5 and 5 points respectively.

[1] Courtesy of Professor Andy Liu.