International Mathematics
# TOURNAMENT OF THE TOWNS

**Senior A-Level Paper** **Spring 2012[1].**

1. In a team of guards, each is assigned a different positive integer. For any two guards, the ratio of the two numbers assigned to them is at least 3:1. A guard assigned the number $n$ is on duty for $n$ days in a row, off duty for $n$ days in a row, back on duty for $n$ days in a row, and so on. The guards need not start their duties on the same day. Is it possible that on any day, at least one in such a team of guards is on duty?

2. One hundred points are marked inside a circle, with no three in a line. Prove that it is possible to connect the points in pairs such that all fifty lines intersect one another inside the circle.

3. Let $n$ be a positive integer. Prove that there exist integers $a_1, a_2, \ldots, a_n$ such that for any integer $x$, the number $(\cdots(((x^2 + a_1)^2 + a_2)^2 + \cdots)^2 + a_{n-1})^2 + a_n$ is divisible by $2n - 1$.

4. Alex marked one point on each of the six interior faces of a hollow unit cube. Then he connected by strings any two marked points on adjacent faces. Prove that the total length of these strings is at least $6\sqrt{2}$.

5. Let $\ell$ be a tangent to the incircle of triangle $ABC$. Let $\ell_a$, $\ell_b$ and $\ell_c$ be the respective images of $\ell$ under reflection across the exterior bisector of $\angle A$, $\angle B$ and $\angle C$. Prove that the triangle formed by these lines is congruent to $ABC$.

6. We attempt to cover the plane with an infinite sequence of rectangles, overlapping allowed.

   (a) Is the task always possible if the area of the $n$th rectangle is $n^2$ for each $n$?

   (b) Is the task always possible if each rectangle is a square, and for any number $N$, there exist squares with total area greater than $N$?

7. Konstantin has a pile of 100 pebbles. In each move, he chooses a pile and splits it into two smaller ones until he gets 100 piles each with a single pebble.

   (a) Prove that at some point, there are 30 piles containing a total of exactly 60 pebbles.

   (b) Prove that at some point, there are 20 piles containing a total of exactly 60 pebbles.

   (c) Prove that Konstantin may proceed in such a way that at no point, there are 19 piles containing a total of exactly 60 pebbles.

**Note:** The problems are worth 4, 5, 6, 6, 8, 3+6 and 6+3+3 points respectively.

[1]Courtesy of Professor Andy Liu

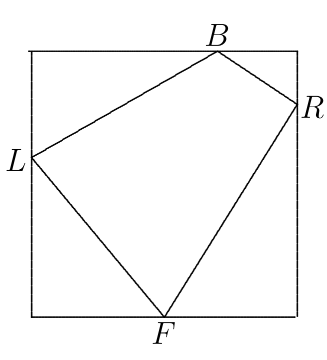
**Solution to Senior A-Level Spring 2012**

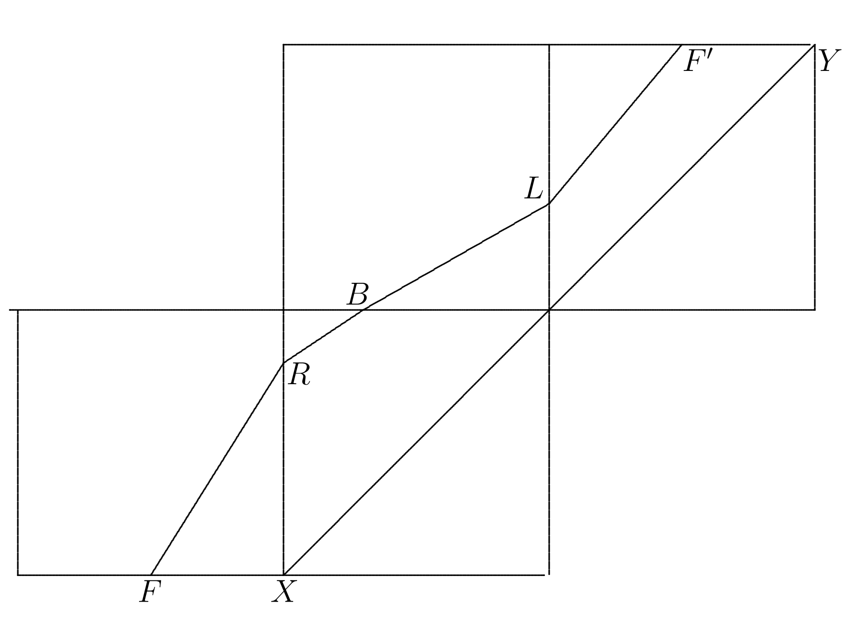
1. Let the guards be $G_1,\ G_2,\ \ldots,\ G_k$ and let $n_1 > n_2 > \cdots > n_k \geq 1$ be the numbers assigned to them. In fact, $n_i \geq 3n_{i+1}$ for $1 \leq i < k$. There is an interval of $3n_2$ days during which $G_1$ is not on duty. Within this interval, there is a subinterval of $n_2 \geq 3n_3$ days during which $G_2$ is not on duty either. Repeating this argument until we reach $G_k$, we will have an interval of $n_k$ days in which none of the guards are on duty.

2. Among all the ways of connecting the one hundred points in pairs, consider the one for which the total length of the fifty segments is maximum, We claim that this connection has the desired property. Suppose to the contrary that two lines, $AB$ and $CD$, intersect outside the circle. Then these four points form a convex quadrilateral, and we may assume that it is $ABCD$. Let $AC$ intersect $BD$ at $E$. Then $AC+BD = AE+BE+CE+DE > AB+CD$. Replacing $AB$ and $CD$ by $AC$ and $BD$ increases the total length of the fifty segments. This contradiction justifies our claim.

3. Note that $1^2 \equiv (2n-2)^2,\ 2^2 \equiv (2n-3)^2,\ \ldots,\ (n-1)^2 \equiv n^2 \pmod{2n-1}$. We claim that for any $i$ and $j$, $1 \leq i < j \leq n-1$, we can find $k$ such that $(i+k)^2 \equiv (j+k)^2 \pmod{2n-1}$. Suppose $j - i = 2m - 1$ for some $m$. Choose $k$ so that $j + k \equiv n + m - 1$ and $i + k \equiv n - m$. Suppose $j - i$ is even. Then $(2n-1) + i - j$ is odd and we can make a similar choice for $k$. This justifies the claim. Now $x^2$ takes on $n$ different values modulo $2n-1$. By a suitable choice of $a_1$, we can make $(x^2 + a_1)^2$ take on at most $n-1$ different values modulo $2n-1$. By a suitable choice of $a_2$, we can make $((x^2+a_1)^2+a_2)^2$ take on at most $n-2$ different values modulo $2n-1$. Continuing in this manner, we can eventually choose $a_{n-1}$ so that $(\cdots(((x^2+a_1)^2+a_2)^2+\cdots)^2+a_{n-1})^2$ takes on only one value. By a suitable choice of $a_n$, we can make the final expression divisible by $2n-1$.

4. Let the points Alex marked be $F$ on the front, $B$ at the back, $R$ to the right, $L$ to the left, $U$ on the up face and $D$ on the down face. The twelve strings formed three closed loops $FRBL$, $FUBD$ and $RULD$. We claim that the total length of each loop is at least $2\sqrt{2}$. Let $FRBL$ be projected onto the down face. Then each point lies on one side of a unit square, as shown in the diagram below on the left. We now fold the loop out as shown in the diagram below on the right. Since $FXYF'$ is a parallelogram, the total length of the strings $FR$, $RB$. $BL$ and $LF'$ is at least $XY$. This is twice the diagonal of a unit square, which is $2\sqrt{2}$.



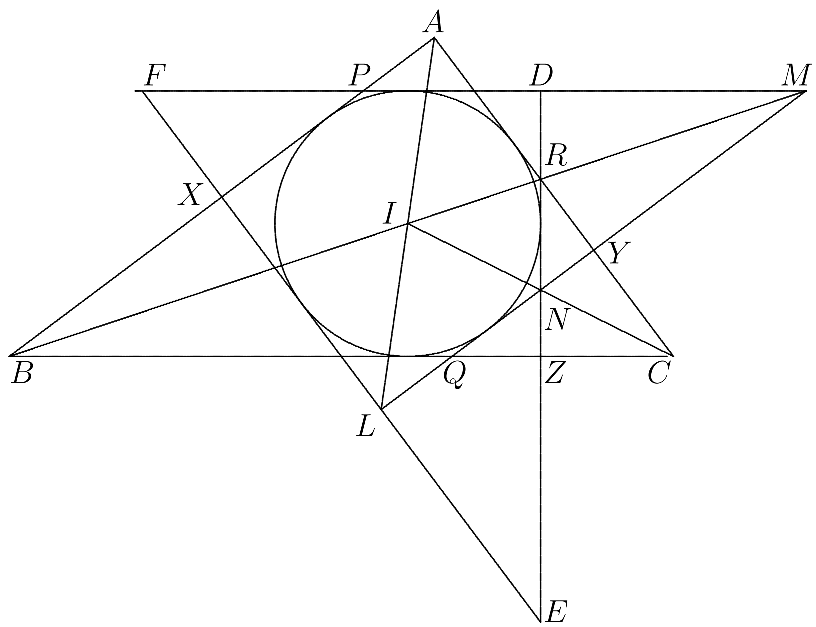
5. Instead of reflecting $\ell$ across the exterior bisectors of the angles of triangle $ABC$, we reflect it across the interior bisectors of these angles. Let $\ell$ intersect the lines $AI$, $BI$ and $CI$ at $L$, $M$ and $N$ respectively, where $I$ is the incentre of $ABC$. Let $\ell$ intersect $BC$ at $Q$ and $CA$ at $Y$. Let $\ell_a$ intersect $\ell_b$ at $F$, $\ell_c$ at $E$ and $AB$ at $X$. Let $\ell_b$ intersect $\ell_c$ at $D$ and $AB$ at $P$. Let $\ell_c$ intersect $BC$ at $Z$ and $CA$ at $R$. Note that $R$ may or may not lie on $BM$. By symmetry about $BM$, $\angle BQM = \angle BPM$. By symmetry about $CN$, $\angle CQN = \angle CRN$. It follows that

$$\angle ARD = \angle CRN = \angle CQN = 180^\circ - \angle BQM = 180^\circ - \angle BPM = \angle APD.$$

Hence $ADRP$ is cyclic, so that $\angle FDE = \angle CAB$. By symmetry about $AL$, $\angle AXL = \angle AYL$. By symmetry about $CN$, $\angle CYN = \angle CZN$. It follows that

$$\angle BZE = \angle CZN = \angle CYN = 180^\circ - \angle AYL = 180^\circ - \angle AXL = \angle BXE.$$

Hence $BEZX$ is cyclic, so that $\angle DEF = \angle ABC$. It follows that triangles $ABC$ and $DEF$ are similar. Now the triangle obtained by reflecting $\ell$ across the interior bisectors of the angles of triangle $ABC$ is clearly similar to $DEF$, and hence to $ABC$. Since these two triangles have the same incircle, they are in fact congruent.




6. (a) **Solution by Daniel Spivak:**
Let the dimensions of the $n$-th rectangle be $n^2 2^n \times \frac{1}{2^n}$. We claim that this sequence of rectangles cannot even cover a disk with radius 1. The intersection of the $n$-th rectangle with the disk is contained in a $2 \times \frac{1}{2^n}$ rectangle and has area less than $\frac{1}{2^{n-1}}$. The total area of these intersections is less than $1 + \frac{1}{2} + \frac{1}{4} + \cdots < 2 < \pi$.

(b) **Solution by Hsin-Po Wang:**
Suppose there exists a positive number $a$ such that the side length of infinitely many of the squares in the sequence is at least $a$. Then we divide the plane into a sequence of $a \times a$ squares in an outward spiral, and these squares can be covered one at a time. Henceforth, we assume that for for positive real number $a$, the number of squares in the sequence with side length at least $a$ is finite. This induces a well-ordering on the squares of the sequence in non-ascending order of side lengths $a_1 \geq a_2 \geq a_3 \geq \cdots$. We may assume that $a_1 < 1$.

We divide the plane into a sequence of unit squares in an outward spiral, and try to cover these squares one at a time. Place the $a_1 \times a_1$ square at the bottom left corner of the first unit square. Place the $a_2 \times a_2$ square on the bottom edge of this unit square to the right of the $a_1 \times a_1$ square. In this manner, we can cover the bottom edge of the unit square because $a_1 + a_2 + \cdots + a_{k_1} > a_1^2 + a_2^2 + \cdots + a_{k_1}^2 \geq 1$ for some $k_1$. Let $b_1 = a_{k_1}$. Then we have covered the bottom strip of the unit square of height $b_1$. We now focus on the $1 \times (1 - b_1)$ rectangle, and apply the same process to cover the bottom strip of height $b_2$ with squares of side lengths $a_{k_1+1},\ a_{k_1+2},\ \ldots,\ a_{k_2} = b_2$.Continuing in this manner, we cover strips of height $b_3,\ b_4,\ \ldots$. We claim that $b_1 + b_2 + \cdots + b_h \geq 1$ for some $h$. Suppose this is not so. Then for all $h$,

$$\begin{aligned} 1 &> (b_1 + b_2 + \cdots + b_h)^2 \\ &\geq b_1(a_{k_1+1} + a_{k_1+2} + \cdots + a_{k_2}) \\ &\quad + b_2(a_{k_2+1} + a_{k_2+2} + \cdots + a_{k_3}) + \cdots \\ &\quad + b_h(a_{k_h+1} + a_{k_h+2} + \cdots + a_{k_{h+1}}) \\ &\geq a_{k_1+1}^2 + a_{k_1+2}^2 + \cdots + a_{k_{h+1}}^2. \end{aligned}$$

This is a contradiction since the last expression is not bounded above.

7. (a) **Solution by Daniel Spivak:**

At some point in time, we must have exactly 70 piles. At least 40 of them contain exactly 1 pebble each, as otherwise the total number of pebbles is at least $39 + 2 \times 31 = 101$. Removing these 40 piles leave behind exactly 30 piles containing exactly 60 pebbles among them.

(b) **Solution by Peter Xie:**

We call $k$ piles containing a total of exactly $2k + 20$ pebbles a good collection. We claim that if $k \geq 23$, a good collection contains either 1 pile with exactly 2 pebbles or 2 piles each with exactly 1 pebble. Otherwise, the total number of pebbles in the collection is at least $1 = 3(k-1) = 3k - 2$, which is strictly greater than $2k + 20$ when $k \geq 23$. Now any partition of the original pile into 40 piles results in a good collection with $k = 40$. From this, we can obtain a good collection with $k = 39$ by either removing 1 pile with exactly 2 pebbles or removing 2 piles each with exactly 1 pebble and then subdividing any other pile with at least 2 pebbles. In the same way, we can obtain good collections down to $k = 22$, with a total of exactly 64 pebbles. We claim that there exist 2 or more piles containing a total of exactly 4 pebbles. Suppose this is not the case. If there are no piles with exactly 1 pebble, then the total number of pebbles in the collection is at least $2 + 3 \times 21 > 64$. If there are piles with exactly 1 pebble, then the total is at least $3 + 4 \times 19 > 64$. Thus the claim is justified. We now remove these 4 pebbles, obtaining 60 pebbles in at most 20 piles. Eventual subdivision of these piles will bring the number of piles to 20 while keeping the total number of pebbles at 60.

(c) **Solution by Peter Xie:**

Separate out piles of 3 until we have 32 piles of 3 and are left with 1 pile of 4. Throughout this process, exactly one pile contains a number of pebbles not divisible by 3. If we include this pile, the total cannot be 60. If we exclude this pile, the total of 19 piles of 3 is only 57. We now separate the pile of 4 into 2 piles of 2. Now every pile contains at most 3 pebbles, so that 19 piles can contain at most 57 pebbles. Further separation will not change this situation.