International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2012[1].**

1. In a team of guards, each is assigned a different positive integer. For any two guards, the ratio of the two numbers assigned to them is at least 3:1. A guard assigned the number n is on duty for n days in a row, off duty for n days in a row, back on duty for n days in a row, and so on. The guards need not start their duties on the same day. Is it possible that on any day, at least one in such a team of guards is on duty?

2. One hundred points are marked inside a circle, with no three in a line. Prove that it is possible to connect the points in pairs such that all fifty lines intersect one another inside the circle.

3. Let n be a positive integer. Prove that there exist integers $a_1, a_2, \ldots, a_n$ such that for any integer x, the number $(\cdots(((x^2+a_1)^2+a_2)^2+\cdots)^2+a_{n-1})^2+a_n$ is divisible by $2n-1$.

4. Alex marked one point on each of the six interior faces of a hollow unit cube. Then he connected by strings any two marked points on adjacent faces. Prove that the total length of these strings is at least $6\sqrt{2}$.

5. Let ℓ be a tangent to the incircle of triangle ABC. Let ℓ_a, ℓ_b and ℓ_c be the respective images of ℓ under reflection across the exterior bisector of $\angle A$, $\angle B$ and $\angle C$. Prove that the triangle formed by these lines is congruent to ABC.

6. We attempt to cover the plane with an infinite sequence of rectangles, overlapping allowed.

   (a) Is the task always possible if the area of the nth rectangle is n^2 for each n?

   (b) Is the task always possible if each rectangle is a square, and for any number N, there exist squares with total area greater than N?

7. Konstantin has a pile of 100 pebbles. In each move, he chooses a pile and splits it into two smaller ones until he gets 100 piles each with a single pebble.

   (a) Prove that at some point, there are 30 piles containing a total of exactly 60 pebbles.

   (b) Prove that at some point, there are 20 piles containing a total of exactly 60 pebbles.

   (c) Prove that Konstantin may proceed in such a way that at no point, there are 19 piles containing a total of exactly 60 pebbles.

Note: The problems are worth 4, 5, 6, 6, 8, 3+6 and 6+3+3 points respectively.

[1] Courtesy of Professor Andy Liu.