International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Spring 2012[1].**

1. A treasure is buried under a square of an 8×8 board, Under each other square is a message which indicates the minimum number of steps needed to reach the square with the treasure. Each step takes one from a square to another square sharing a common side. What is the minmum number of squares we must dig up in order to bring up the treasure for sure?

2. The number 4 has an odd number of odd positive divisors, namely 1, and an even number of even positive divisors, namely 2 and 4. Is there a number with an odd number of even positive divisors and an even number of odd positive divisors?

3. In the parallelogram $ABCD$, the diagonal AC touches the incircles of triangles ABC and ADC at W and Y respectively, and the diagonal BD touches the incircles of triangles BAD and BCD at X and Z respectively. Prove that either W, X, Y and Z coincide, or $WXYZ$ is a rectangle.

4. Brackets are to be inserted into the expression $10 \div 9 \div 8 \div 7 \div 6 \div 5 \div 4 \div 3 \div 2$ so that the resulting number is an integer.

   (a) Determine the maximum value of this integer.

   (b) Determine the minimum value of this integer.

5. RyNo, a little rhinoceros, has 17 scratch marks on its body. Some are horizontal and the rest are vertical. Some are on the left side and the rest are on the right side. If RyNo rubs one side of its body against a tree, two scratch marks, either both horizontal or both vertical, will disappear from that side. However, at the same time, two new scratch marks, one horizontal and one vertical, will appear on the other side. If there are less than two horizontal and less than two vertical scratch marks on the side being rubbed, then nothing happens. If RyNo continues to rub its body against trees, is it possible that at some point in time, the numbers of horizontal and vertical scratch marks have interchanged on each side of its body?

Note: The problems are worth 3, 4, 4, 2+3 and 5 points respectively.

[1]Courtesy of Professor Andy Liu.

Solution to Junior O-Level Spring 2012

1. Whichever square we dig up first, there is no guarantee that the treasure is there. If the message we get says that the treasure is one square away, we cannot determine its location uniquely. Thus we have to dig up at least three squares. Let the first two squares we dig up be at the lower left corner and the lower right corner. We may as well suppose that we do not find the treasure under either of them. The diagram below shows the possible coordinates of the treasure based on the messages under these two squares. Since every square has a unique pair of coordinates, the treasure can be brought up by digging up just one more square.

7,14	8,12	9,12	10,11	11,10	12,9	13,8	14,7
6,13	7,12	8,11	9,10	10,9	11,8	12,7	13,6
5,12	6,11	7,10	8,9	9,8	10,7	11,6	12,5
4,11	5,10	6,9	7,8	8,7	9,6	10,5	11,4
3,10	4,9	5,8	6,7	7,6	8,5	9,4	10,3
2,9	3,8	4,7	5,6	6,5	7,4	7,3	9,2
1,8	2,7	3,6	4,6	5,5	6,3	7,2	8,1
0,7	1,6	2,5	3,4	4,3	5,2	6,1	7,0

2. **Solution ny Ling Long:**
All integers under discussion are taken to be positive. The divisors of an integer n can be divided into pairs such that the product of the two numbers in each pair is n, except when n is a square, with $\sqrt{n}$ having no partner. Suppose there exists an integer n with an even number of odd divisors and an odd number of even divisors. Then it has an odd number of divisors in total, and must be a square, say $n = x^2$. Let $x = 2^k y$ where y is odd. Then $n = 2^{2k} y^2$, and the odd divisors of n are precisely the divisors of y^2. This number cannot be even.

3. Suppose $ABCD$ is a rhombus. Then both W and Y coincide with the midpoint of AC, and both X and Z coincide with the midpoint of BD. Since AC and BD bisect each other, all four points coincide. Suppose $ABCD$ is not a rhombus. Then none of the four points coincides with the common midpoint of AC and BD. Hence they are distinct. By symmetry, $AW = CY$ and $BX = DZ$. Hence WY and XZ also bisect each other, so that $WXYZ$ is a parallelogram. Let the incircle of ABC touch AB at P and BC at Q. We may assume that W is closer to A and Y is closer to C. Then

$$WY = CW - CY = CW - AW = CQ - AP = CB - AB.$$

Similarly, $XZ = AD - AB = WY$. Being a parallelogram with equal diagonals, $WXYZ$ is a rectangle.

4. Bracketing simply separates the factors 10, 9, ..., 2 into the numerator and the denominator of the overall expression.

   (a) We have

$$\begin{aligned} & 10 \div (((((((9 \div 8) \div 7) \div 6) \div 5) \div 4) \div 3) \div 2) \\ = \; & 10 \div \frac{9}{8!} \\ = \; & \frac{10!}{9^2} \\ = \; & 44800. \end{aligned}$$

   Since 9 is the second number in the sequence, it must be in the denominator. Hence the maximum value cannot be higher than 44800.

   (b) Since 7 is the only number in the sequence divisible by 7, it must be in the numerator. Hence the minimum value cannot be lower than 7. We have

$$\begin{aligned} & (10 \div 9) \div ((8 \div 7) \div (6 \div ((5 \div 4) \div (3 \div 2)))) \\ = \; & \frac{10}{9} \div \left(\frac{8}{7} \div \left(6 \div \left(\frac{5}{4} \div \frac{3}{2}\right)\right)\right) \\ = \; & \frac{10}{9} \div \left(\frac{8}{7} \div \left(6 \div \frac{6}{5}\right)\right) \\ = \; & \frac{10}{9} \div \left(\frac{8}{7} \div \frac{36}{5}\right) \\ = \; & \frac{10}{9} \div \frac{10}{63} \\ = \; & 7. \end{aligned}$$

5. Let $a,\ b,\ c$ and d be the numbers of scratch marks which are horizontal and on the left side, vertical and on the left side, horizontal and on the right side, and vertical and on the right side. Suppose the initial values of a and b have been interchanged, and so are those of c and d, then $a + b$ and $c + d$ are unchanged. Since each of these two sums changes by 2 after a rubbing, the total number of rubbings must be even. If we allow negative values temporarily, the order of the rubbings is immaterial, and we can assume that they occur alternately on the left side and on the right side. After each pair of rubbings, the parity of each of $a,\ b,\ c$ and d has changed. Suppose initially $a + b$ is odd so that $c + d$ is even. After an odd number of pairs of rubbings, the final values of a and b may have interchanged from their initial values, the odd one becomes even and the even one becomes odd. However, this is not possible for c and d, as they either change from both even to both odd, or from both odd to both even. Similarly, after an even number of pairs of rubbings, the final values of c and d may have interchanged from their initial values, but this is not possible for a and b. Thus the desired scenario cannot occur.