International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2012.**

1. A table 10×10 was filled according to the rules of the game “Bomb Squad”: several cells contain bombs (one bomb per cell) while each of the remaining cells contains a number, equal to the number of bombs in all cells adjacent to it by side or by vertex.

Then the table is rearranged in the “reverse” order: bombs are placed in all cells previously occupied with numbers and the remaining cells are filled with numbers according to the same rule. Can it happen that the total sum of the numbers in the table will increase in a result?

Solution. The answer is no. In a given table consider all unordered pairs of adjacent cells (by side or by vertex) one of which has a bomb and another is empty (two pairs are different it they differ in at least one cell). It is clear that the sum of all numbers in the table equals to the number of these pairs.

For the complementary table (bomb and no-bomb cells reversed) we have the same number of such pairs. Therefore the sum of the numbers in the complementary table will be the same as in the original table.

2. Given a convex polyhedron and a sphere intersecting each its edge at two points so that each edge is trisected (divided into three equal parts). Is it necessarily true that all faces of the polyhedron are

(a) congruent polygons?

(b) regular polygons?

Solution. a) The answer is negative. Consider a regular prism with triangular base and square lateral faces. On each edge, mark the trisecting points. Clearly they are equidistant from the centre of the prism and thus belong to the corresponding sphere.

b) The answer is positive. Suppose $A_1 \dots A_n$ is a face of the polyhedron. All the points B_i, C_i trisecting its sides $A_{i-1}A_i$ lie on the same circle that is the intersection of the sphere with the plane of the face. Suppose $A_{i-1}A_i = 3a$, $A_iA_{i+1} = 3b$. By secant theorem, $A_iB_i \cdot A_iC_i = A_iB_{i+1} = A_iC_{i+1} \Leftrightarrow 2a^2 = 2b^2 \Leftrightarrow a = b$. Hence the face is equilateral. It remains to prove that it has equal angles. All segments B_iC_i are equal as well as isosceles triangles B_iOC_i where O is the centre of the circle. Hence the equality holds for all triangles B_iOA_i, all angles B_iA_iO and all angles $\angle A_{i-1}A_iA_{i+1} = 2\angle B_iA_iO$.

3. For a class of 20 students several field trips were arranged. In each trip at least four students participated. Prove that there was a field trip such that each student who participated in it took part in at least 1/17-th of all field trips.

Solution. Let n be the number of arranged trips. Let us call a student "enthusiastic " if he/she took part in at least $n/17$ of trips, and "ordinary" otherwise. Assume that there was no trip attended by only enthusiastic students. Choose an ordinary student in each trip. The total number of their attendances is at least n, the number of attendances of each ordinary student is less than $n/17$, so the number of chosen students exceeds 17, and the number of enthusiastic students is at most 2. Then the total number of their attendances is at most $2n$, and for the ordinary students this number is less than $n/17 \cdot 20$. So the total number of all attendances is less than $4n$, a contradiction with the condition of the problem. Therefore, there was a trip attended by only enthusiastic students.

4. Let $C(n)$ be the number of prime divisors of a positive integer n.

(a) Consider set S of all pairs of positive integers (a, b) such that $a \neq b$ and

$$C(a + b) = C(a) + C(b)$$

Is S finite or infinite?

(b) Define S' as a subset of S consisting of the pairs (a, b) such that $C(a + b) > 1000$. Is S' finite or infinite?

Solution.

(a) S is infinite. *Example 1*. $a = 2^k$, $b = 2^{k+1}$, $(a+b) = 3 \cdot 2^k$, $k = 1, 2, \ldots$. Then $C(a) = 1$, $C(b) = 1$, $C(a + b) = 2$.

Example 2 (based on different idea). Let $a = p$, $b = 5p$, $(a+b) = 6p = 2 \cdot 3 \cdot p$. Let $p \neq 2, 3, 5$ is a prime. Then $C(a) = 1$, $C(b) = 2$, $C(a + b) = 3$.

(b) S' is infinite. Let $p_1, p_2, \ldots, p_k$ be the first prime numbers, where $k > 1000$.

Then we can write $p_1 p_2 \ldots p_k - 1 = P_1^{r_1} P_2 \ldots P_s^{r_s}$, where $P_i \neq p_j$ are also primes and $0 < s < k$. Let $m = k - s$ and let us choose primes $q_1, q_2, \ldots, q_m$, so that $q_i \neq P_j$ and $q_i \neq p_l$. Finally, consider a and b of the form $a = q_1 q_2 \ldots q_m$, $b = q_1 q_2 \ldots q_m P_1^{r_1} P_2 \ldots P_s^{rs}$, so that $(a + b) = q_1 q_2 \ldots q_m p_1 p_2 \ldots p_k$. Then $C(a) = m$, $C(b) = m + s$, $C(a + b) = m + k$ and $C(a + b) = C(a) + C(b)$ is equivalent to $m + s = k$. Since there is an infinite number of primes we can always choose an infinite number of different $q_1, q_2, \ldots, q_m$ and therefore create an infinite number of different pairs (a, b) satisfying the conditions.

5. Among 239 coins identical in appearance there are two counterfeit coins. Both counterfeit coins have the same weight different from the weight of a genuine coin.

Using a simple balance, determine in three weighings whether the counterfeit coin is heavier or lighter than the genuine coin. A simple balance shows if both sides are in equilibrium or left side is heavier or lighter. It is not required to find the counterfeit coins.

Solution (one of possible versions). Split coins in three groups A (80), B (80), C (79). On Step 1: we weigh A **?** B.

(1) If $A = B$ then either both A and B consist of genuine coins or there is one counterfeit coin in each group. Split A into the groups A_1 (40) and A_2 (40). On Step 2: A_1 **?** A_2.

  (a) If $A_1 = A_2$ then A_1 consists of real coins (so does B) and therefore both counterfeits are in the group C. Choose 79 coins from the group $A + B$; let it be A'. On Step 3: C **?** A':

    A. If $C > A'$ then the counterfeit coin is heavier and

    B. If $C < A'$ the counterfeit coin is lighter.

  (b) If $A_1 \neq A_2$, say $A_1 > A_2$, then both groups A and B contain a counterfeit coin and therefore all 79 coins in the group C are genuine.
  Choose 40 coins from the group C; let it be the group C'. On Step 3: A_1 **?** C'.

    A. If $A_1 = C'$ then the counterfeit coin is lighter and

    B. if $A_1 > C'$ then the counterfeit coin is heavier.
    The case $A_1 < C'$ is impossible.

(2) Let $A \neq B$, say $A > B$. Split A into the groups A_1 (40) and A_2 (40). On Step 2: A_1 **?** A_2.

  i. If $A_1 \neq A_2$ then the counterfeit coin is heavier, and the third weighing is not needed.

  ii. If $A_1 = A_2$ then either both groups contain a single counterfeit coin or all coins in these groups are genuine. Split A_1 into $A_11(20)$ and $A_12(20)$. On Step 3: A_11 **?** A_12.

    A. If $A_11 = A_12$ then all coins in A_11, A_12 and A_2 are genuine, B contains one or two counterfeit coins and these coins are lighter than genuine ones.

    B. If $A_11 \neq A_12$ then A contains a counterfeit coin, B does not, and counterfeit coins are heavier than genuine ones.