TOURNAMENT OF TOWNS

Junior O-Level Paper **Fall 2012.**

1. Five students have the first names Clark, Donald, Jack, Robin and Steve, and have the last names (in a different order) Clarkson, Donaldson, Jackson, Robinson and Stevenson. It is known that Clark is 1 year older than Clarkson, Donald is 2 years older than Donaldson, Jack is 3 years older than Jackson, Robin is 4 years older than Robinson.

Who is older, Steve or Stevenson and what is the difference in their ages?

Solution. The sum of ages of Clark, Donald, Jack, Robin and Steve is equal to the sum of ages of Clarkson, Donaldson, Jackson, Robinson and Stevenson. Hence Stevenson is older than Steve, and the difference is $1 + 2 + 3 + 4 = 10$ years.

2. Let $C(n)$ be the number of prime divisors of a positive integer n. (For example, $C(10) = 2$, $C(11) = 1$, $C(12) = 2$).

Consider set S of all pairs of positive integers (a, b) such that $a \neq b$ and

$$C(a + b) = C(a) + C(b).$$

Is set S finite or infinite?

Solution. The set of pairs is infinite. *Example 1*. $a = 2^k$, $b = 2^{k+1}$, $(a + b) = 3 \cdot 2^k$, $k = 1, 2, \ldots$. Then $C(a) = 1$, $C(b) = 1$, $C(a + b) = 2$.

Example 2 (based on different idea). Let $a = p$, $b = 5p$, $(a+b) = 6p = 2 \cdot 3 \cdot p$. Let $p \neq 2, 3, 5$ is a prime. Then, $C(a) = 1$, $C(b) = 2$, $C(a + b) = 3$.

3. A table 10×10 was filled according to the rules of the game "Bomb Squad": several cells contain bombs (one bomb per cell) while each of the remaining cells contains a number, equal to the number of bombs in all cells adjacent to it by side or by vertex.

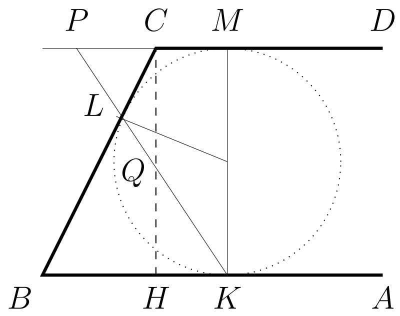
Then the table is rearranged in the "reverse" order: bombs are placed in all cells previously occupied with numbers and the remaining cells are filled with numbers according to the same rule. Can it happen that the total sum of the numbers in the table will increase in a result?

Solution. The answer is no. In a given table consider all unordered pairs of adjacent cells (by side or by vertex) one of which has a bomb and another is empty (two pairs are different it they differ in at least one cell). It is clear that the sum of all numbers in the table equals to the number of these pairs.

For the complementary table (bomb and no-bomb cells reversed) we have the same number of such pairs. Therefore the sum of the numbers in the complementary table will be the same as in the original table.

4. A circle touches sides AB, BC, CD of a parallelogram $ABCD$ at points K, L, M respectively. Prove that the line KL bisects the height of the parallelogram drawn from the vertex C to AB.

Solution. Let CH be the height dropped from the vertex C to the side AB. Let P and Q be the points of intersection of line KL with CD and CH respectively.



Observe that $BK = BL$ and $CL = CM$ (as tangents to a circle) and that triangles BLK and PCL are similar (angle–angle criteria). Therefore the triangle LPC is isosceles, $CP = CL$ and therefore $PC = CM$. Hence, CH is the midline in triangle PMK, so $CQ = MK/2 = CH/2$.

5. For a class of 20 students several field trips were arranged. In each trip at least one student participated. Prove that there was a field trip such that each student who participated in it took part in at least 1/20-th of all field trips.

Solution. Let n be the number of trips. Let us call a student "enthusiastic " if he/she took part in at least than $n/20$ of trips, and an "ordinary" student otherwise.

Let k_i be the number of trips that i-th ordinary student participated ($i \leq 20$). Then the sum of attendances of the ordinary students is $k_1 + k_2 + \cdots < 20 \times n/20 = n$. Since the number of trips is n, there was a trip free of ordinary students.