International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2011.**

1. Baron Munchausen has a set of 50 coins. The mass of each is a distinct positive integer not exceeding 100, and the total mass is even. The Baron claims that it is not possible to divide the coins into two piles with equal total mass. Can the Baron be right?

2. In the coordinate space, each of the eight vertices of a rectangular box has integer coordinates. If the volume of the solid is 2011, prove that the sides of the rectangular box are parallel to the coordinate axes.

3. (a) Does there exist an infinite triangular beam such that two of its cross-sections are similar but not congruent triangles?

   (b) Does there exist an infinite triangular beam such that two of its cross-sections are equilateral triangles of sides 1 and 2 respectively?

4. There are n red sticks and n blue sticks. The sticks of each colour have the same total length, and can be used to construct an n-gon. We wish to repaint one stick of each colour in the other colour so that the sticks of each colour can still be used to construct an n-gon. Is this always possible if

   (a) $n = 3$;

   (b) $n > 3$?

5. In the convex quadrilateral $ABCD$, BC is parallel to AD. Two circular arcs ω_1 and ω_3 pass through A and B and are on the same side of AB. Two circular arcs ω_2 and ω_4 pass through C and D and are on the same side of CD. The measures of ω_1, ω_2, ω_3 and ω_4 are α, β, β and α respectively. If ω_1 and ω_2 are tangent to each other externally, prove that so are ω_3 and ω_4.

6. In every cell of a square table is a number. The sum of the largest two numbers in each row is a and the sum of the largest two numbers in each column is b. Prove that $a = b$.

7. Among a group of programmers, every two either know each other or do not know each other. Eleven of them are geniuses. Two companies hire them one at a time, alternately, and may not hire someone already hired by the other company. There are no conditions on which programmer a company may hire in the first round. Thereafter, a company may only hire a programmer who knows another programmer already hired by that company. Is it possible for the company which hires second to hire ten of the geniuses, no matter what the hiring strategy of the other company may be?

Note: The problems are worth 4, 6, 3+4, 4+4, 8, 8 and 11 points respectively.