International Mathematics
TOURNAMENT OF THE TOWNS

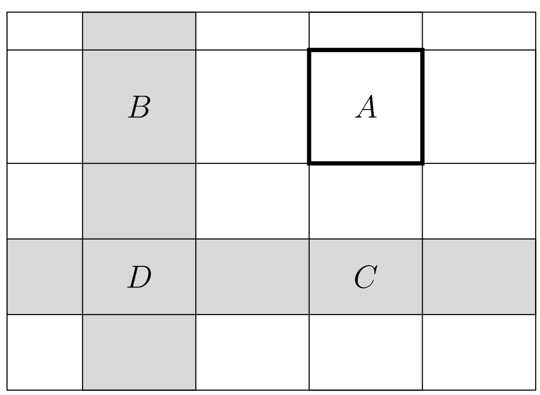
Solution to Junior O-Level Spring 2011

1. Suppose that we managed to place the numbers on a circle so that the difference between two adjacent numbers is odd. This means that odd and even numbers must alternate. From the condition follows that each number has both neighbours either both greater or both less than itself.

   Note that 1 is an odd number and it can only have the neighbours greater than itself. Since the numbers increase and decrease alternately, each odd number has the neighbours greater than itself and therefore each even number has the neighbours less than itself. However number 2 can have possibly only one number less than itself. Contradiction.

2. The number of cells with non integer perimeters is at most $121 - 111 = 10$. No matter how these cells are distributed in the rectangle, at least one row and one column consist of cells with integer perimeters. On the figure below these raw and column are shaded.

   We prove that any other cell has an integer perimeter. Let A be this shell and (a, b) be the dimensions its dimensions. Then it has a perimeter $p = 2a+2b = (2a+2c)+(2b+2d)-(2c+2d)$ where $(2a+2c)$, $(2b+2d)$, and $(2c+2d)$ are the perimeters of cells C, B, and D respectively. Since all these perimeters are integers, the perimeter $2a + 2b$ is also integer.



3. Yes, it is possible. (Actually one can grow any number of worms in one hour).

   We can take for granted that in time t the worm grows additional t in the length $(0 < t \leq 1)$.

   Assume that at moment 0 there was 1 fully grown worm. Let dissect it into two parts of lengths t and $(1 - t)$ respectively, $0 < t \leq \frac{1}{2}$.

   Then, at moment t the part that had the length t becomes $2t$, while another part becomes fully grown worm. We dissect it into parts of the lengths $(2t)$ and $(1 - 2t)$. respectively. Therefore, after the dissection we have two worms of length $2t$ and one of the length $(1 - 2t)$.

   In time $2t$ after the last dissection (or $t + 2t = 3t$) from the beginning, we have two worms of length $4t$ (each grown from the part $2t$) and one fully grown worm. Again, we dissect the fully grown this time into the parts of length $4t$ and $1-4t$. After this dissection we have three worms of length $4t$ and one of length $1-4t$.

Similarly, in time $4t = 2^2$ ($3t + 4t = 7t$ from the beginning) we have three worms of lengh $8t = 2^3t$ and one fully grown. Again, we dissect the last one into the parts of length $8t$ and $1-8t$. At that moment we have 4 worms of length $8t$ and one of length $(1-8t)$.

It is easy to see the pattern:

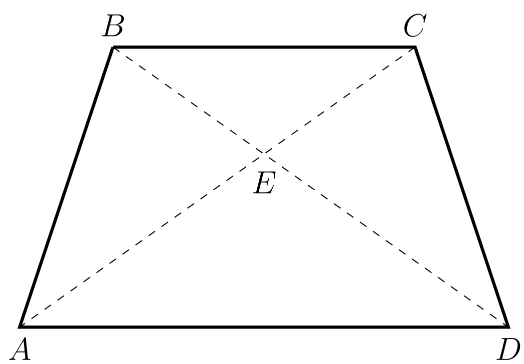
In time $2^k - 1$ from the beginning we have $k+1$ worms of length $2^k t$ and one fully grown. If we want at some moment to have all worms be fully grown, we should take t: $2^k t = 1$ or $t = 1/2^k$ and at this moment we would have $k+2$ fully grown worms.

To have 10 fully grown worms in less than one hour (let us say in 1/2 of hour) we can choose $t = 1/2^9 = 1/512$.

4. Consider isosceles trapezoid $ABCD$, $AD \parallel BC$, $AB = BC = CD$, $\angle BAD = \angle CDA = 72°$.

   In virtue due to $AB = BC = CD$ triangles ABC and BCD are isosceles. Note that $\angle ABC = \angle BCD = 180° - 72° = 108°$. Since triangle ABC is isosceles, $\angle BAC = \angle BCA = (180° - 72°)/2 = 36°$. Then $\angle ACD = 108° - 36° = 72°$; but we know that $\angle CDA = 72°$ and therefore triangle ACD is isosceles (and so does triangle BDA). So, each diagonal divides $ABCD$ into two isosceles triangles.

   Obviously, triangles BEC and AED are isosceles. Note that $\angle AEB$ as an exterior angle of triangle AED is equal to $\angle EAD + \angle ADE = 36° + 35° = 72°$ which is equal to $\angle ABE$. Therefore triangle ABE is isosceles and so is triangle CED. So, both diagonals divide $ABCD$ into four isosceles triangles.



5. On the first day let the knight split the coins in piles of 25 and 75 coins and each following day move one more coin from the bigger pile to the smaller one. Then in 25 days (if not earlier) the knight sets free.

   Really, let m and n respectively be the numbers of magical and usual coins in the small pile on the first day ($m + n = 25$, $0 < m < 25$). In the worst case scenario (that takes the longest number of days to set free) in 24 days (first day including) in the smaller pile will be 48 coins, 24 magical and 24 usual coins. Therefore, on the next day, when either magical or usual coin is moved, the number of magical or usual coins becomes 25. Therefore, one kind of coins is split in half among the piles and the knight sets free.