International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring 2011.**

1. Does there exist a hexagon that can be divided into four congruent triangles by a straight cut?

2. Passing through the origin of the coordinate plane are 180 lines, including the coordinate axes, which form 1° angles with one another at the origin. Determine the sum of the x-coordinates of the points of intersection of these lines with the line $y = -x + 100$.

3. Baron Munchausen has a set of 50 coins. The mass of each is a distinct positive integer not exceeding 100, and the total mass is even. The Baron claims that it is not possible to divide the coins into two piles with equal total mass. Can the Baron be right?

4. Given an integer $n > 1$, prove that there exist distinct positive integers a, b, c and d such that $a + b = c + d$ and $\frac{a}{b} = \frac{nc}{d}$.

5. AD and BE are altitudes of an acute triangle ABC. From D, perpendiculars are dropped to AB at G and AC at K. From E, perpendiculars are dropped to AB at F and BC at H. Prove that FG is parallel to HK and $FK = GH$.

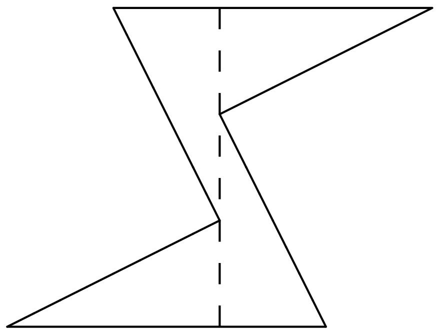
6. Two ants crawl along the sides of the 49 squares of a 7×7 board. Each ant passes through all 64 vertices exactly once and returns to its starting point. What is the smallest possible number of sides covered by both ants?

7. In every cell of a square table is a number. The sum of the largest two numbers in each row is a and the sum of the largest two numbers in each column is b. Prove that $a = b$.

Note: The problems are worth 4, 4, 5, 6, 7, 10 and 10 points respectively.

Solution to Junior A-Level Spring 2011

1. The diagram below shows such a hexagon and how it is cut.

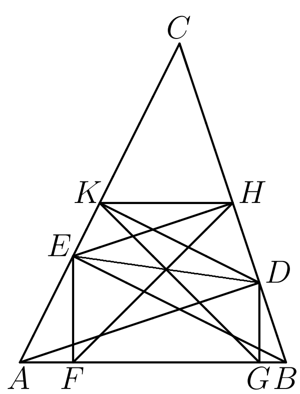


2. The line $x+y=100$ is parallel to the line making a $135°$ angle with the positive x-axis. Hence there are only 179 points of intersection. The point of intersection of $x+y=100$ with $y=x$ is (50,50), and the other points of intersections are arranged in symmetric pairs with respect to this point. Hence the desired sum is $179 \times 50 = 8950$.

3. The Baron is right, as usual. The masses of the 50 coins in his collection may just be the 50 even numbers up to 100. Their total mass is 50 times 51, so that each of two piles would have total mass 25 times 51, which is odd. This is impossible since all the coins have even weights.

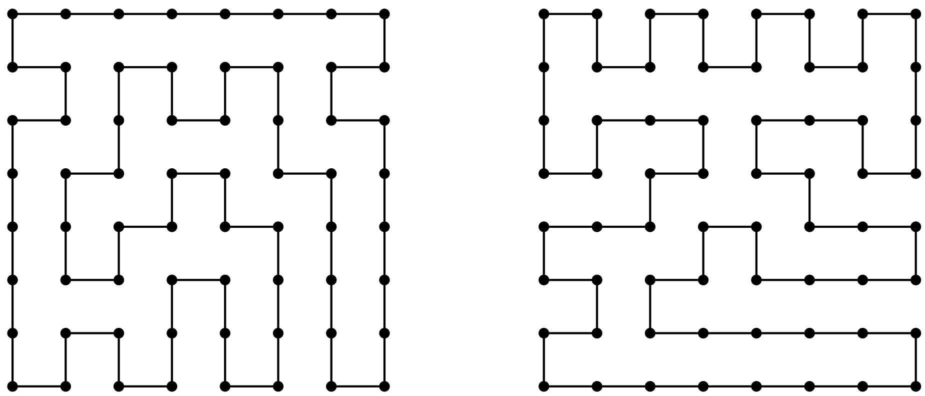
4. To satisfy $\frac{a}{b}=\frac{nc}{d}$ only, we can take $a=2n,\ b=1,\ c=2$ and $d=1$. Note that $a+b=2n+1$ while $c+d=3$. To satisfy $a+b=c+d$ as well, we change these values to $a=3(2n)=6n$, $b=3(1)=3,\ c=(2n+1)2=4n+2$ and $d=(2n+1)1=2n+1$. These four numbers are distinct when $n>1$.

5. Since $\angle EHD=90°=\angle EKD$, $EDHK$ is a cyclic quadrilateral. Hence $\angle EHK=\angle EDK$. Now DK and BE are parallel since both are perpendicular to AC. Hence $\angle EDK=\angle DEB$. Finally, $\angle DEB=\angle DAB$ since $ABDE$ is also a cyclic quadrilateral. Hence $\angle EHK=\angle DAB$. Now HK and AB make equal angles with the parallel lines EH and DA. Hence HK is parallel to AB and therefore to FG.



Since $\angle EFB=90°=\angle EHB$, $EFBH$ is a cyclic quadrilateral. Hence $\angle EHF=\angle EBA$. Similarly, $\angle DKG=\angle DAB$. Hence $\angle FHK=\angle FHE+\angle EHK=\angle EBA+\angle DAB$. By symmetry, $\angle GKH=\angle GKD+\angle DKH=\angle DAB+\angle EBA$ also. It follows easily that triangles FHK and GKH are congruent, so that $FK=GH$.

6. The total number of sides is $2 \times 7 \times 8 = 112$ and each ant covers 64 sides. Hence the number of sides covered by both ants cannot be less than $2 \times 64 - 112 = 16$. The following diagram shows the paths of the two ants with exactly 16 sides covered by both, every other side along the four edges of the board.



7. **First Solution by Daniel Spivak.**
We only need to prove that $a \geq b$ since we will then have $b \geq a$ by symmetry. Circle the largest number x_j in column j, $1 \leq j \leq n$, where n is the number of rows and therefore of columns. By relabelling if necessary, we may assume that x_1 is the smallest of these n numbers. We consider two cases.
Case 1. Two circled numbers x_j and x_k are on the same row.
Then $a \geq x_j + x_k \geq 2x_1 \geq b$ since the sum of the largest two numbers in column 1 is b.
Case 2. Each circled number is in a different row.
Let the second largest number y_1 in column 1 be in row j, and let the circled number in row j be x_k. Then $b = x_1 + y_1 \leq x_k + y_1 \leq a$ since the sum of the largest two numbers in row k is a.

Second Solution:
Suppose to the contrary that $a \neq b$. We may assume by symmetry that $a > b$. Circle in each row the largest two numbers. Let the number of circled numbers in column i be c_i, $1 \leq i \leq n$, where n is the number of rows and therefore of columns. We have $c_1 + c_2 + \cdots + c_n = 2n$. Now $\binom{c_1}{2} + \binom{c_2}{2} + \cdots + \binom{c_n}{2} = \frac{1}{2}(c_1^2 + c_2^2 + \cdots + c_n^2) - \frac{1}{2}(c_1 + c_2 + \cdots + c_n)$. By the Root-Mean-Square Inequality, $\sqrt{\frac{c_1^2 + c_2^2 + \cdots + c_n^2}{n}} \geq \frac{c_1 + c_2 + \cdots + c_n}{n}$. It follows that $\binom{c_1}{2} + \binom{c_2}{2} + \cdots + \binom{c_n}{2} \geq n$. We now construct a graph with n vertices representing the n rows. Two vertices are joined by an edge if and only if the corresponding rows have circled numbers in the same column. The number of edges of this graph is given by $\binom{c_1}{2} + \binom{c_2}{2} + \cdots + \binom{c_n}{2} \geq n$, so that the graph has a cycle, say of length k. By relabelling if necessary, the k vertices on this cycle represent rows 1 to k, with the circled numbers on row i in columns i and $i + 1$ for $1 \leq i \leq k - 1$, and the circled numbers on row k in columns k and 1. Now the circled numbers in a column may or may not be the largest two of its numbers, but the sum of these $2k$ numbers is ka. This means that the sum of the largest two numbers of some column is at least $a > b$, which is a contradiction.