International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Fall 2011.**

1. Pete has marked at least 3 points in the plane such that all distances between them are different. A pair of marked points A and B will be called *unusual* if A is the furthest marked point from B, and B is the nearest marked point to A (apart from A itself). What is the largest possible number of unusual pairs that Pete can obtain?

2. Let $0 < a, b, c, d < 1$ be real numbers such that $abcd = (1-a)(1-b)(1-c)(1-d)$. Prove that $(a+b+c+d) - (a+c)(b+d) \geq 1$.

3. In triangle ABC, points D, E and F are bases of altitudes from vertices A, B and C respectively. Points P and Q are the projections of F to AC and BC respectively. Prove that the line PQ bisects the segments DF and EF.

4. Does there exist a convex n-gon such that all its sides are equal and all vertices lie on the parabola $y = x^2$, where

   (a) $n = 2011$;

   (b) $n = 2012$?

5. We will call a positive integer *good* if all its digits are nonzero. A good integer will be called *special* if it has at least k digits and their values are strictly increasing from left to right. Let a good integer be given. In each move, one may insert a special integer into the digital expression of the current number, on the left, on the right or in between any two of the digits. Alternatively, one may also delete a special number from the digital expression of the current number.What is the largest k such that any good integer can be turned into any other good integer by a finite number of such moves?

6. Prove that for $n > 1$, the integer $1^1 + 3^3 + 5^5 + \ldots + (2^n - 1)^{2^n - 1}$ is a multiple of 2^n but not a multiple of 2^{n+1}.

7. A blue circle is divided into 100 arcs by 100 red points such that the lengths of the arcs are the positive integers from 1 to 100 in an arbitrary order. Prove that there exist two perpendicular chords with red endpoints.

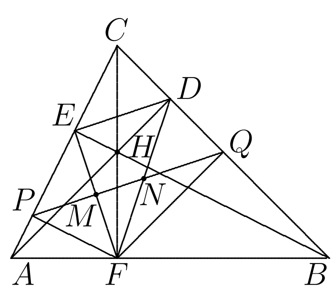
Note: The problems are worth 4, 4, 5, 3+4, 7, 7 and 9 points respectively.

Solution to Senior A-Level Fall 2011

1. First, we show by example that we may have one unusual pair when there are at least 3 points. Let A and B are chosen arbitrarily. Add some points within the circle with centre B and radius AB but outside the circle with centre A and radius AB. Then B is the point nearest to A while A is the point furthest from B. We now prove that if there is another pair of unusual points, we will have a contradiction. We consider two cases.
Case 1.
The additional unusual pair consists of C and D such that D is the point nearest to C while C is the point furthest from D. Now $DA > AB$ since B is the point nearest to A, $AB > BC$ because A is the point furthest from B, $BC > CD$ since D is the point nearest to C, and $CD > DA$ because C is the point furthest from D. Hence $DA > DA$, which is a contradiction.
Case 2.
The additional unusual pair consists of B and C such that C is the point nearest to B and B is the point furthest from C. Now $CA > AB$ since B is the point nearest to A, $AB > BC$ since A is the point furthest from B, and $BC > CA$ because B is the point furthest from C. Hence $CA > CA$, which is a contradiction.

2. **Solution by Adrian Tang:**
From $abcd = (1-a)(1-b)(1-c)(1-d)$, we have $\frac{ac}{(1-a)(1-c)} = \frac{(1-b)(1-d)}{bd}$. It follows that $\frac{a+c-1}{(1-a)(1-c)} = \frac{ac}{(1-a)(1-c)} - 1 = \frac{(1-b)(1-d)}{bd} - 1 = \frac{1-b-d}{bd}$. Now $\frac{(a+c-1)(1-b-d)}{(1-a)(1-c)bd} \geq 0$ since it is the product of two equal terms. From $(1-a)(1-c)bd > 0$, we have $(a+c-1)(1-b-d) \geq 0$. Expansion yields $a-ab-ad+c-cb-cd-1+b+d \geq 0$, which is equivalent to $a+b+c+d-(a+c)(b+d) \geq 1$.

3. **Solution by Wen-Hsien Sun:**
Let H be the orthocentre of triangle ABC. Then H is the incentre of triangle DEF. Note that $CDHE$ and $CQFP$ are cyclic quadrilaterals. Hence $\angle ADE = \angle FCP = \angle FQP$. Since AD and FQ are parallel to each other, so are ED and PQ. Thus $\angle PQD = \angle EDC = \angle FDQ$. Let M and N be the points of intersection of PQ with EF and DF respectively. Then $ND = NQ$. Since $\angle FQD = 90°$, N is the circumcentre of triangle FQD so that $FN = ND$. Similarly, we have $FM = ME$.



4.

(a) This is possible. Let V be the vertex of the parabola. On one side, mark on the parabola points $A_1, A_2, \ldots, A_{1005}$ such that $VA_1 = A_1A_2 = \cdots = A_{1004}A_{1005} = t$, and on the other side, points $B_1, B_2, \ldots, B_{1005}$ such that $VB_1 = B_1V_2 = \cdots = B_{1004}B_{1005} = t$. The length ℓ of $A_{1005}B_{1005}$ varies continuously with t. When t is very small, we have $\ell > t$. When t is very large, ℓ is less than a constant times $\sqrt{t}$, which is in turn less than t. Hence at some point in between, we have $\ell = t$.

(b) We first prove a geometric result.

Lemma.

In the convex quadrilateral $ABCD$, $AB = CD$ and $\angle ABC + \angle BCD > 180°$. Then $AD > BC$.

Proof:

Complete the parallelogram $BCDE$. Then $\angle EBC + \angle BCD = 180° < \angle ABC + \angle BCD$. Hence $\angle ABC > \angle EBC$. We also have $BD = BD$ and $AB = CD = EB$. By the Side-angle-side Inequality, $AD > ED + BC$.

Corollary.

If A, B, C and D are four points in order on a parabola ad $AB = CD$, then $AD > BC$.

Proof:

Since the parabola is a convex curve, the extension of AB and DC meet at some point L, Then $\angle ABC + \angle BCD = (\angle ALD + \angle BCL) + (\angle ALD + \angle CBL) = 180° + \angle ALD > 180°$. We will now apply the It follows from the Lemma that $AD > BC$.

Returning to our problem, suppose P_1, P_2, $\dots$, P_{2012} are points in order on a parabola, such that $P_1P_2 = P_2P_3 = \cdots = P_{2011}P_{2012}$. We now apply the Corollary 1005 times to obtain $P_1P_{2012} > P_2P_{2011} > \cdots > P_{1006}P_{1007}$. Hence the 2012-gon cannot be equilateral.

5. **Solution by Daniel Spivak:**

We cannot have $k = 9$ as the only special number would be 123456789. Adding or deleting it does not change anything. We may have $k = 8$. We can convert any good number into any other good number by adding or deleting one digit at a time. We give below the procedures for adding digits. Reversing the steps allows us to deleting digits.

Adding 1 or 9 anywhere.

Add 123456789 and delete 23456789 or 12345678.

Adding 2 anywhere.

Add 23456789 and then add 1 between 2 and 3. Now delete 13456789.

Adding 8 anywhere.

Add 12345678 and then add 9 between 7 and 8. Now delete 12345679.

Adding 3 anywhere.

Add 23456789 and delete 2. Now add 1 and 2 between 3 and 4 and delete 12456789.

Adding 7 answhere.

Add 12345678 and delete 8. Now add 8 and 9 between 6 and 7 and delete 12345689.

Adding 4 anywhere.

Add 23456789 and delete 2 and 3, Now add 1, 2 and 3 between 4 and 5 and delete 12356789.

Adding 6 anywhere.

Add 12345678 and delete 7 and 8. Now add 7, 8 and 9 between 5 and 6 and delete 12345789.

Adding 5 anywhere.

Add 23456789, delete 2, 3 and 4 and add 1, 2, 3 and 4 betwen 5 and 6. Albernately, add 12345678, delete 6, 7 and 8 and add 6, 7, 8 and 9 between 4 and 5. Now delete 12346789.

6.

We first prove two preliminary results.

Lemma 1.

For any positive odd integer k, $k^{2^n} \equiv 1 \pmod{2^{n+2}}$.

Proof:
We have $k^{2^n} - 1 = (k-1)(k+1)(k^2+1)(k^4+1)\cdots(k^{2^{n-1}}+1)$. This is a product of $n+1$ even factors. Moreover, one of $k-1$ and $k+1$ is divisible by 4. The desired result follows.
Lemma 2.
For any integer $n \geq 2$, $(2^n+k)^k \equiv k^k(2^n+1) \pmod{2^{n+2}}$.
Proof:
Expanding $(2^n+k)^k$, all the terms are divisible by 2^{n+2} except have $\binom{k}{1}k^{k-1}2^n$ and k^k. The desired result follows.
Let the given sum be denoted by S_n. We now use induction on n to prove that S_n is divisible by 2^n but not 2^{n+1} for all $n \geq 2$. Note that $S_2 = 1^1 + 3^3 = 28$ is divisible by 2^2 but not by 2^3. Using Lemmas 1 and 2, we have

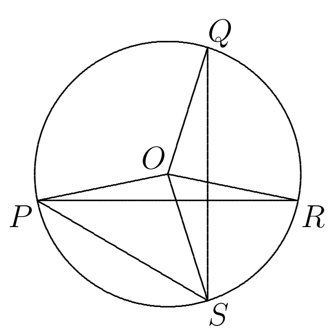
$$\begin{aligned} S_{n+1} - S_n &= \sum_{i=1}^{2^{n-1}} (2^n + 2i - 1)^{2^n+2i-1} \\ &\equiv \sum_{i=1}^{2^{n-1}} (2^n + 2i - 1)^{2i-1} \pmod{2^{n+2}} \\ &\equiv \sum_{i=1}^{2^{n-1}} (2i-1)^{2i-1}(2^n+1) \pmod{2^{n+2}} \\ &= S_n(2^n+1). \end{aligned}$$

Now $S_{n+1} = 2S_n(2^{n-1}+1)$. By the induction hypothesis, S_n is divisible by 2^n but not by 2^{n+1}. It follows that S_{n+1} is divisible by 2^{n+1} but not by 2^{n+2}.

7.

We first prove a geometric result.
Lemma.
Let P, Q, R and S be four points on a circle in cyclic order. If the arcs PQ and RS add up to a semicircle, then PR and QS are perpendicular chords.

Proof:
Let O be the centre of the circle. From the given condition, $\angle POQ + \angle ROS = 180°$. Hence $\angle PSQ + \angle RPS = \frac{1}{2}(\angle POQ + \angle ROS) = 90°$. It follows that PR and QS are perpendicular chords.



The circumference of the circle is 5050. By an arc is meant any arc with two red endpoints. A simple arc is one which contains no red points in its interior. A compound arc is one which consists of at least two adjacent simple arcs whose total lengths is less than 2525. For each red point P, choose the longest compound arc with P as one of its endpoints. We consider two cases.

Case 1.

Suppose for some P there are two equal choices in opposite directions. Then they must be of length 2475, and the part of the circle not in either of them is the simple arc of length 100. It follows that one of these two compound arcs, say PQ, does not contain the simple arc RS of length 50. We may assume that P, Q, R and S are in cyclic order. If $Q = R$, then PS is a diameter of the circle. If $S = P$, then QR is a diameter of the circle. In either case, PQ and RS are perpendicular chords. If these four points are distinct, then the arcs PQ and RS add up to a semicircle. By the Lemma, PR and QS are perpendicular chords.

Case 2.

The choice is unique for every P. Then we have chosen at least 50 such arcs since each may serve as the maximal arc for at most two red points. Each of these arcs has length at least 2476 but less than 2525, a range of 49 possible values. By the Pigeonhole Principle, two of them have the same length $2525 - k$ for some k where $1 \leq k \leq 49$. If one of them does not contain the simple arc $\mathcal{C}$ of length k, we can then argue as in Case 1. Suppose both of them contain $\mathcal{C}$. This is only possible if one of them ends with $\mathcal{C}$ and the other starts with it. By removing $\mathcal{C}$ from both, we obtain two disjoint compund acrs both of length $2525 - 2k$, and note that $2 \leq 2k \leq 98$. Now one of them does not contain the simple arc of length $2k$, and we can argue as in Case 1.