International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2011.**

1. An integer $n > 1$ is written on the board. Alex replaces it by $n + d$ or $n - d$, where d is any divisor of n greater than 1. This is repeated with the new value of n. Is it possible for Alex to write on the board the number 2011 at some point, regardless of the initial value of n?

2. P is a point on the side AB of triangle ABC such that $AP = 2PB$. If $CP = 2PQ$, where Q is the midpoint of AC, prove that ABC is a right triangle.

3. A set of at least two objects with pairwise different weights has the property that for any pair of objects from this set, we can choose a subset of the remaining objects so that their total weight is equal to the total weight of the given pair. What is the minimum number of objects in this set?

4. A game is played on a board with 2012 horizontal rows and $k > 2$ vertical columns. A marker is placed in an arbitrarily chosen cell of the left-most column. Two players move the marker in turns. During each move, the player moves the marker one cell to the right, or one cell up or down to a cell that has never been occupied by the marker before. The game is over when any of the players moves the marker to the right-most column. There are two versions of this game. In Version A, the player who gets the marker to the right-most column wins. In Version B, this player loses. However, it is only when the marker reaches the second column from the right that the players learn whether they are playing Version A or Version B. Does either player have a winning strategy?

5. Let $0 < a, b, c, d < 1$ be real numbers such that $abcd = (1-a)(1-b)(1-c)(1-d)$. Prove that $(a + b + c + d) - (a + c)(b + d) \geq 1$.

6. A car goes along a straight highway at the speed of 60 kilometres per hour. A 100 metre long fence is standing parallel to the highway. Every second, the passenger of the car measures the angle of vision of the fence. Prove that the sum of all angles measured by him is less than $1100°$.

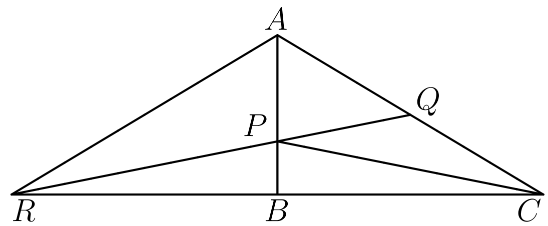
7. Each vertex of a regular 45-gon is red, yellow or green, and there are 15 vertices of each colour. Prove that we can choose three vertices of each color so that the three triangles formed by the chosen vertices of the same color are congruent to one another.

Note: The problems are worth 3, 4, 5, 6, 6, 7 and 9 points respectively.

Solution to Junior A-Level Fall 2011

1. **Solution by Wen-Hsien Sun:**
Starting from any positive integer n, Alex adds n a total of 2010 times to get $2011n$. Then he subtracts 2011 a total of $n-1$ times to get 2011.

2. Extend QP to R so that $RP = 2PQ$. Then P is the centroid of triangle ARC. Since $AP = 2PB$, the extension of AP intersects RC at its midpoint B. Since $RP = CP$, triangles PRB and PCB are congruent, so that $\angle ABR = \angle ABC$. Since their sum is $180°$, each is $90°$.



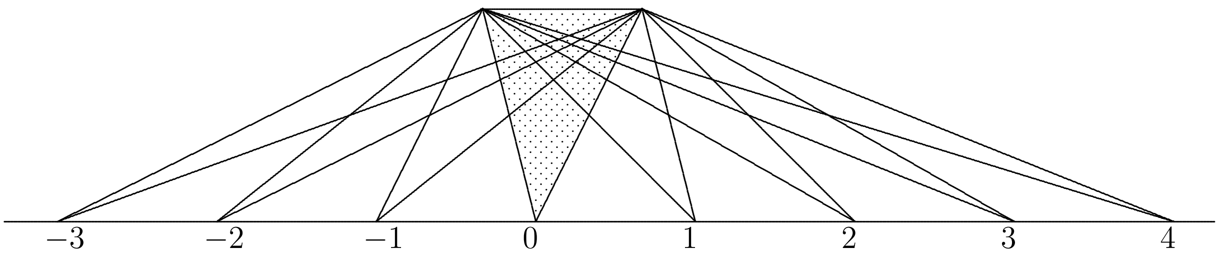
3. Clearly, the set cannot have 2, 3 or 4 objects, as it would not be possible to balance the heaviest two objects. Suppose it has only 5 objects, of respective weights $a > b > c > d > e$. Clearly, we must have $a+b = c+d+e$. Since $a+c > b+d$, we must also have $a+c = b+d+e$, which implies $b = c$. The set may have 6 objects, of respective weights 8, 7, 6, 5, 4 and 3. Then 8+7=6+5+4, 8+6=7+4+3, 8+5=7+6, 8+4=7+5, 8+3=7+4=6+5, 7+3=6+4, 6+3=5+4, 5+3=8 and 4+3=7.

4. The first player has a winning strategy. She will only move the counter up or down until she learns whether Version A or Version B of the game is being played. Since 2012 is even, she can choose a direction (up or down) so that the marker stays in the same column for an odd number of moves. If the marker starts at the end of the column, the only possible direction allows her to keep it in the same column for an odd number of moves. Thus she can ensure that the second player is always the one to move the marker to the right, whether by choice earlier or being forced to do so when the marker reaches the end of the column. When the marker reaches the second column from the right, if Version A is being played, the first player can win by simply moving the marker to the right. If Version B is being played, she can keep the marker in this column as before, and wait for the second player to lose.

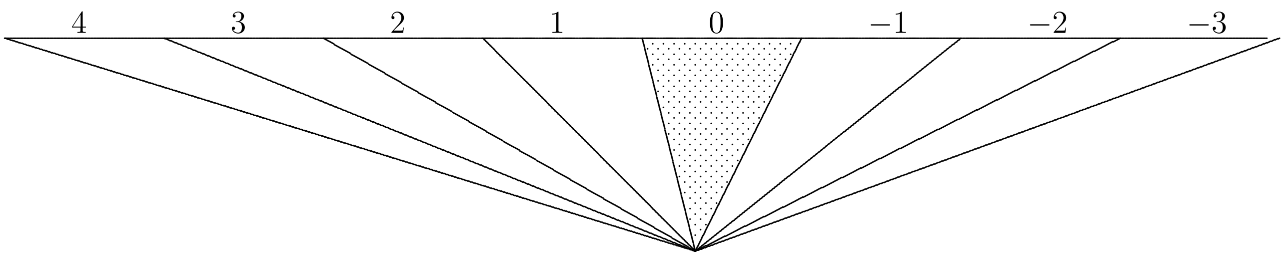
5. From $abcd = (1-a)(1-b)(1-c)(1-d)$, we have

$$\begin{aligned}
& a+b+c+d-(a+c)(b+d) \\
= \; & 1 + ac(1-b-d) + bd(1-a-c) \\
= \; & 1 + ac(1-b)(1-d) + bd(1-a)(1-c) - 2abcd \\
\geq \; & 1 + 2\sqrt{ac(1-b)(1-d)bd(1-a)(1-c)} - 2abcd \\
= \; & 1 + 2abcd - 2abcd \\
= \; & 1.
\end{aligned}$$

6. **Solution by Central Jury:**
Divide the points of observation into six groups cyclically, so that the points in each group are 100 metres apart, the same as the length of the billboard.The diagram below shows the angles of visions from the points of a group.



We now parallel translate all these points to a single point, along with their billboards and angles of vision, as shown in the diagram below.



The sum of all these angles is clearly at most $180°$. Since there are six groups of points of observations, the sum of all angles of vision is at most $6 \times 180° < 1100°$.

7. **Solutioin by Central Jury:**
Copy the regular 45-gon onto a piece of transparency and marked on it the 15 red points. Call this the Red position, and rotate the piece of transparency about the centre of the 45-gon $8°$ at a time. For each of the 45 positions, count the number of matches of yellow points with the 15 marked points. Since each of the 15 yellow points may match up with any of the 15 marked points, the total number of matches is $15 \times 15 = 225$, so that the average number of matches per position is 5. However, in the Red position, the number of matches is 0. Hence there is a position with at least 6 matches. Call this the Yellow position, choose any 6 of the matched marked points and erase the other 9. Repeat the rotation process, but this time counting the number of matches of green points with the 6 marked points. The total number of matches is $6 \times 15 = 90$, so that the average number of matches per position is 2. As before, there is a position with at least 3 matches. Call this the Green position, choose any 3 of the matched marked points and erase the other 3. The 3 remaining marked points define three congruent triangles, a red one in the Red position, a yellow one in the Yellow position and a green one at the Green position.