International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Spring 2010.**[1]

1. Bananas, lemons and pineapples are being delivered by 2010 ships. The number of bananas in each ship is equal to the total number of lemons in the other 2009 ships, and the number of lemons in each ship is equal to the total number of pineapples in the other 2009 ships. Prove that the total number of fruit being delivered is a multiple of 31.

2. Each line in the coordinate plane has the same number of common points with the parabola $y = x^2$ and with the graph $y = f(x)$. Prove that $f(x) = x^2$.

3. Is it possible to cover the surface of a regular octahedron by several regular hexagons, without gaps or overlaps?

4. Baron Münchausen claims that a polynomial $P(x)$ with non-negative integers as coefficients is uniquely determined by the values of $P(2)$ and $P(P(2))$. Surely the Baron is wrong, isn't he?

5. A segment is given on the plane. In each move, it may be rotated about either of its endpoints in a $45°$ angle clockwise or counterclockwise. Is it possible that after a finite number of moves, the segment returns to its original position except that its endpoints are interchanged?

Note: The problems are worth 3, 4, 5, 5 and 6 points respectively.

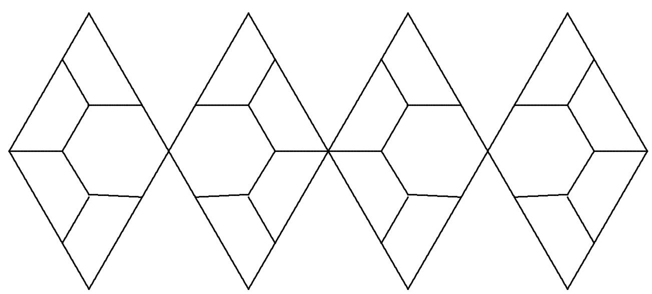
[1]Courtesy of Andy Liu.

Solution to Senior O-Level Spring 2010

1. In counting the total number of lemons, we have counted each pineapple 2009 times. Hence the total number of lemons is 2009 times the total number of pineapples. Similarly, in counting the total number of bananas, we have counted each lemon 2009 times. Hence the total number of bananas is 2009 times the total number of lemons, and $2009^2 = 4036081$ times the total number of pineapples. It follows that the total number of fruit is equal to the total number of pineapples times $1 + 2009 + 4036081 = 4038091 = 31 \times 130261$, and is therefore a multiple of 31.

2. Note that $f(x)$ is uniquely defined for all x since it is given to be a function. In any case, since $y = x^2$ intersects each vertical line in exactly one point, so does $y = f(x)$. Let S be the region of the plane below $y = x^2$. Every point in S lies on a line which does not intersect $y = x^2$. Hence no point of $y = f(x)$ can belong to S. For any real number r, consider the line tangent to $y = x^2$ at the point (r, r^2). Except for this point, the line lies entirely in S. Since this line intersects $y = f(x)$ at exactly one point, we must have $f(r) = r^2$. Since r is an arbitrary real number, $f(x) = x^2$.

3. It is possible to accomplish the task with twelve hexagons, as shown in the diagram below.



4. **Solution by Central Jury.**
Let $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$, where the coefficients are non-negative integers. Suppose $P(2) = b$. Then $b = a_0 2^n + a_1 2^{n-1} + \cdots + a_n > a_0 + a_1 + \cdots + a_n$. It follows that we have $b^n > b^{n-1}(a_0 + a_1 + \cdots + a_n) \geq a_1 b^{n-1} + \cdots + a_{n-1}b + a_n$. Now $\frac{P(b)}{b^n} = a_0 + \frac{a_1 b^{n-1} + \cdots + a_{n-1}b + a_n}{b*n}$. Then $a_0 = \lfloor \frac{P(b)}{b^n} \rfloor$, where n is the largest integer for which $P(b) \geq b^n$. In an analogous manner, $a_1 = \lfloor \frac{P(b) - a_0 b^n}{b^{n-1}} \rfloor$, and so on. It follows that $P(x)$ is uniquely determined, and the Baron is right!
Remark:
The values of $P(2)$ and $P(P(2))$ cannot be assigned arbitrarily. Suppose we have $P(2) = 13$ and $P(13) = 2224$, the above algorithm yields $n = 3$, $a_0 = \lfloor \frac{2224}{13^3} \rfloor = 1$, $a_1 = \lfloor \frac{2224 - 13^3}{13^2} \rfloor = 0$, $a_2 = \lfloor \frac{2224 - 2197}{13} \rfloor = 2$ and $a_3 = 27 - 2 \times 13 = 1$. On the other hand, if $P(2) = 3$ and $P(3) = 5$, we get $n = 1$, $a_0 = \lfloor \frac{5}{3} \rfloor = 1$ and $a_1 = 5 - 3 = 2$, but $P(x) = x + 2$ yields $P(2) = 4$. This is because the correct polynomial $P(x) = 2x - 1$ does not satisfy the hypothesis of the problem, and the above algorithm cannot be applied.

5. **Solution by Jonathan Zung.**
Suppose the task is possible. Let the segment be of length 1. Label one of its endpoints A and the other B. We combine consecutive moves making rotations about the same point into one, so that the new moves alternately rotate about A and B through an angle which is a multiple of 45°. Denote the initial positions of A and B by A_0 and B_0 respectively. By symmetry, we may assume that the first rotation is about B. Denote the new position of A by A_1. The next rotation is about A_1. Denote the new position of B by B_1. Continue until $A_k = B_0$ or $B_k = A_0$ for some k. We may assume that the former is the case. Then we have a $(2k-1)$-gon $A_1B_1 \ldots A_k$ whose edges are all of length 1 and may intersect one another. Each horizontal edge represents a horizontal displacement of 1 unit, while each slanting edge represents a horizontal displacement of $\frac{1}{\sqrt{2}}$ unit. These are incommensurable. In going around the perimeter of the polygon once, the net horizontal displacement is 0. Hence we must have an even number of horizontal edges and an even number of slanting edges. Similarly, we must also have an even number of vertical edges. Hence the total number of edges of this polygon must be even, but a $(2k-1)$-gon has an odd number of edges. This is a contradiction.