International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2010.**[1]

1. Is it possible to divide the lines in the plane into pairs of perpendicular lines so that every line belongs to exactly one pair?

2. Alex has a piece of cheese. He chooses a positive number α and cut the piece into two, in the ratio $1 : \alpha$. He can then choose any piece and cut it in the same way. Is it possible for him to obtain, after a finite number of cuts, two piles of pieces each containing half the original amount of cheese, if
(a) α is irrational;
(b) $\alpha \neq 1$ is rational?

3. Can we obtain the number 2010 from the number 1 by applying any combination of the functions sin, cos, tan, cot, arcsin, arccos, arctan and arccot?

4. At a convention, each of the 5000 participants watched at least one movie. Several participants can form a discussion group if either they had all watched the same movie, or each had watched a movie nobody else in the group had. A single participant may also form a group. Prove that the number of groups could always be exactly 100.

5. On a circular road are 33 horsemen, riding in the same direction, each at a different constant speed. There is only one point along the road at which a horseman is allowed to pass another horseman. Can they continue to ride for an arbitrarily long period?

6. A circle with centre I is tangent to all four sides of a convex quadrilateral $ABCD$. M and N are the midpoints of AB and CD respectively. If $\frac{IM}{AB} = \frac{IN}{CD}$, prove that $ABCD$ has a pair of parallel sides.

7. A multi-digit number is written on the blackboard. Susan puts in a number of plus signs between some pairs of adjacent digits. The addition is performed and the process is repeated with the sum. Prove that regardless of what number was initially on the blackboard, Susan can always obtain a single-digit number in at most ten steps.

Note: The problems are worth 3, 2+2, 6, 6, 7, 8 and 9 points respectively.

[1]Courtesy of Andy Liu.

Solution to Senior A-Level Spring 2010

1. Form families consisting of all mutually parallel lines. Put into a group two families whose lines are perpendicular. For each group, choose an arbitrary line ℓ not parallel to either family. Each line in a family intersects exactly one point of ℓ, and each point of ℓ lies on exactly one line in the family. Thus each point of ℓ defined one line from each family, and these two lines form a pair. This procedure may be applied to all groups, so that every line in the plane is in exactly one pair.

2. **Solution by Central Jury.**

   (a) Let $\alpha < 1$. The first cut creates the piece $\frac{1}{1+\alpha}$ and $\frac{\alpha}{1+\alpha}$. Then cut the larger piece into $\frac{1}{(1+\alpha)^2}$ and $\frac{\alpha}{(1+\alpha)^2}$. We want $\frac{1}{(1+\alpha)^2} = \frac{\alpha}{(1+\alpha)^2} + \frac{\alpha}{1+\alpha}$ or $1 = \alpha + \alpha(1+\alpha)$. Solving $\alpha^2 + 2\alpha - 1 = 0$, we have $\alpha = \frac{-2\pm\sqrt{4+4}}{2} = -1 \pm \sqrt{2}$. Since $\alpha > 0$, $\alpha = \sqrt{2} - 1$.

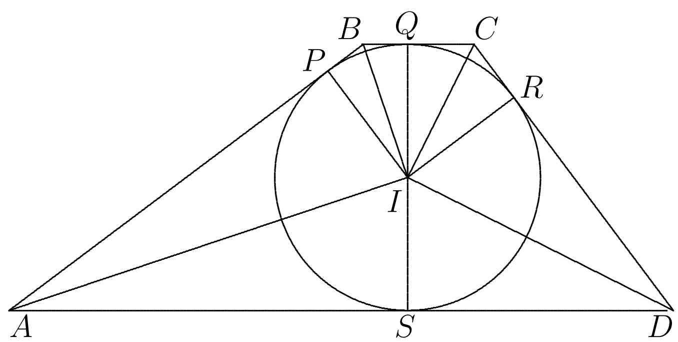
   (b) Let $\alpha = \frac{m}{n}$ where m and n are relatively prime positive integers. In the first step, we have the pieces $\frac{m}{m+n}$ and $\frac{n}{m+n}$. In all subsequent steps, we will cut all pieces. There is no harm in assuming this since the two parts of a piece which is not to be cut can just stay together. Suppose the task is accomplished after k steps. Each of the 2^k pieces is $\frac{m^i n^{k-i}}{(m+n)^k}$ for $0 \le i \le k$, with $i = 0$ occuring only once. Each numerator is a multiple of m except for n^k. Thus the division into two piles of equal amount is not possible.

3. **Solution by Zhi Qiang Liu.**
   If $x = \frac{1}{\sqrt{n}}$, then $\arctan x$ is an angle θ in a right triangle with opposite side 1 and adjacent side $\sqrt{n}$. By Pythagoras' Theorem, the length of the hypotenuse is $\sqrt{n+1}$, so that $\sin\theta = \frac{1}{\sqrt{n+1}}$. Define $f(x) = \sin(\arctan x)$. Starting with $1 = \frac{1}{\sqrt{1}}$, we can apply $f(x)$ repeatedly and obtain $\frac{1}{\sqrt{2010^2}} = \frac{1}{2010}$. Now $\cot(\arctan\frac{1}{2010}) = 2010$.

4. **Solution by Zhi Qiang Liu and Cristina Rosu, independently.**
   We construct united groups in the first stage and diverse groups in the second stage. In the k-th step of the first stage, we create a united group of size at least $101-k$. This stage terminates, perhaps even immediately, when no such groups can be formed. If this is as a result of having nobody left, then we have formed at most 100 groups since $1+2+\cdots+100 > 5000$. We can take individuals out of existing groups to form groups of one until we have exactly 100 groups. Suppose after the n-th step, we cannot form a united group of size at least $101-(n+1)$ from the remaining participants. We proceed to the second stage. We have created n united groups, and now we create $100-n$ diverse groups. Start with any movie watched by at least one of these participants. There are less than $100-n$ of them, and they can be put into separate groups. This will be the movie each of them has watched that nobody else in their group would have. Take another movie watched by at least one of the remaining participants. There are less than $100-n$ of them, including some who are already in the groups. Those that are not can now join groups not including anyone who has watched this movie. The remaining participants can be added to the groups in an analogous manner. If some of these $100-n$ groups happen to be empty, we can take individuals out out of existing groups to form groups of one until we have exactly 100 groups.

5. **Solution by Jonathan Zung.**
We use induction on the number n of runners. For $n = 1$, there is nothing to prove. Suppose the result holds for some $n \geq 1$, each with a distinct integer speed. Let M be the least common multiple of these speeds. If we add an $(n+1)$-st runner with speed 0 at the passing point, the result still holds. Now increase the speed of each of the $n+1$ runners by M. Since their relative speeds remain the same, the result continues to hold. In particular, it holds for $n = 33$.

6. **Solution by Jonathan Zung.**
Let P, Q, R and S be the points of tangency of the circle with AB, BC, CD and DA respectively. Let $\angle AIS = \angle AIP = \alpha$, $\angle BIP = \angle BIQ = \beta$, $\angle CIQ = \angle CIR = \gamma$ and $\angle DIR = \angle DIS = \delta$. Then $\angle AIB + \angle CID = \alpha + \beta + \gamma + \delta = 180°$. If $\angle AIB > 90°$, then $\angle CID < 90°$. The point I will be inside the circle with AB as diamter but outside the circle with CD as diameter. Hence $\frac{IM}{AB} < \frac{1}{2} < \frac{IN}{CD}$. Similarly, if $\angle AIB < 90°$, then $\frac{IM}{AB} > \frac{1}{2} > \frac{IN}{CD}$. Both contradict the hypothesis that $\frac{IM}{AB} = \frac{IN}{CD}$. Hence $\alpha + \beta = \angle AIB = 90°$ so that Q, I and S are collinear. Since both BC and DA are perpendicular to QS, they are parallel to each other.



7. **Solution by Jonathan Zung.**
We use an overline to denote the concatenation of digits. Let the given number be $\overline{a_0a_1a_2\dots a_n}$ and the sum of the digits be S. Suppose $S \leq 10^{10}$. By putting a plus sign between every pair of adjacent digits in each step, Susan obtain a number with at most 11 digits in the first step, a number at most 99 in the second step, a number at most 18 in the third step and a single-digit number on the fourth step. Suppose $S > 10^{10}$. Define $A = \overline{a_0a_1a_2} + \overline{a_3a_4a_5} + \overline{a_6a_7a_8} + \cdots$, $B = a_0 + \overline{a_1a_2a_3} + \overline{a_4a_5a_6} + \overline{a_7a_8a_9} + \cdots$ and $C = \overline{a_0a_1} + \overline{a_2a_3a_4} + \overline{a_5a_6a_7} + \cdots$. Note that $A + B + C > 100S$ so that one of A, B and C is greater than $10S$. By symmetry, we may assume it is B. Then there exists a positive integer t such that $S < 10^t < B$. Consider now the following sequence of numbers.

$$\begin{aligned} S &= a_0 + a_1 + a_2 + a_3 + \cdots, \\ & \quad a_0 + \overline{a_1a_2a_3} + a_4 + a_5 + \cdots, \\ & \quad a_0 + \overline{a_1a_2a_3} + \overline{a_4a_5a_6} + a_7 + \cdots, \\ & \quad \cdots \\ B &= a_0 + \overline{a_1a_2a_3} + \overline{a_4a_5a_6} + \overline{a_7a_8a_9} + \cdots. \end{aligned}$$

These numbers are increasing by steps of less than 1000. Hence one of them will be at least 10^t and at most $\min\{10^t + 999, B\}$. This will be the first step for Susan, arriving at a number with at most four non-zero digits. By putting a plus sign between every pair of adjacent digits in each subsequent step, Susan obtain a number at most 36 in the second step, a number at most 11 in the third step and a single-digit number on the fourth step.