International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2009**

1 [**3**] Is it possible to split all straight lines in a plane into the pairs of perpendicular lines, so that every line belongs to a single pair?

2 Alex has a piece of cheese. He chooses a positive number a and cuts the piece into several pieces one by one. Every time he choses a piece and cuts it in the same ratio $1 : a$. His goal is to divide the cheese into two piles of equal masses. Can he do it if

- **(a)** [**2**] a is irrational?
- **(b)** [**2**] a is rational, $a \neq 1$?

3 [**6**] Consider a composition of functions sin, cos, tan, cot, arcsin, arccos, arctan, arccot, applied to the number 1. Each function may be applied arbitrarily many times and in any order. (ex: sin cos arcsin cos sin ...1). Can one obtain the number 2010 in this way?

4 [**6**] 5000 movie fans gathered at a convention. Each participant had watched at least one movie. The participants should be split into discussion groups of two kinds. In each group of the first kind, the members would discuss a movie they all watched. In each group of the second kind, each member would tell about the movie that no one else in this group had watched. Prove that the chairman can always split the participants into exactly 100 groups. (A group consisting of one person is allowed; in this case this person submits a report).

5 [**7**] 33 horsemen are riding in the same direction along a circular road. Their speeds are constant and pairwise distinct. There is a single point on the road where the horsemen can surpass one another. Can they ride in this fashion for arbitrarily long time ?

6 [**8**] Quadrilateral $ABCD$ is circumscribed around the circle with centre I. Let points M and N be the midpoints of sides AB and CD respectively and let $IM/AB = IN/CD$. Prove that $ABCD$ is either a trapezoid or a parallelogram.

7 [**9**] Peter writes some positive integer on a blackboard. Susan can place pluses between some of its digits; then the children calculate the resulting sum (for example, starting from 123456789 one may obtain $12345 + 6 + 789 = 13140$). Susan is allowed to apply this procedure to the resulting number up to ten times. Prove that she can always end up with one-digit number.