

**International Mathematics
TOURNAMENT OF THE TOWNS**

Junior A-Level Paper

Spring 2010.¹

1. Alex has a piece of cheese. He chooses a positive number $\alpha \neq 1$ and cut the piece into two, in the ratio $1 : \alpha$. He can then choose any piece and cut it in the same way. Is it possible for him to obtain, after a finite number of cuts, two piles of pieces each containing half the original amount of cheese?
2. M is the midpoint of the side CA of triangle ABC . P is some point on the side BC . AP and BM intersect at the point O . If $BO = BP$, determine $\frac{OM}{PC}$.
3. Along a circle are placed 999 numbers, each 1 or -1 , and there is at least one of each. The product of each block of 10 adjacent numbers along the circle is computed. Let S denote the sum of these 999 products.
 - (a) What is the minimum value of S ?
 - (b) What is the maximum value of S ?
4. Is it possible that the sum of the digits of a positive integer n is 100 while the sum of the digits of the number n^3 is 100^3 ?
5. On a circular road are N horsemen, riding in the same direction, each at a different constant speed. There is only one point along the road at which a horseman is allowed to pass another horseman. Can they continue to ride for an arbitrarily long period if
 - (a) $N = 3$;
 - (b) $N = 10$?
6. A broken line consists of 31 segments joined end to end. It does not intersect itself, and has distinct end points. What is the smallest number of straight lines which can contain all segments of such a broken line?
7. A number of fleas are on a 10×10 chessboard, each in a different cell. Every minute, a flea jumps to the adjacent square either to the east, to the south, to the west or to the north. It continues to jump in the same direction as long as this is possible, but reverses direction if it has reached the edge of the chessboard. In one hour, no two fleas ever occupy the same cell. What is the maximum number of fleas on the chessboard?

Note: The problems are worth 3, 4, 3+3, 6, 3+5, 8 and 11 points respectively.

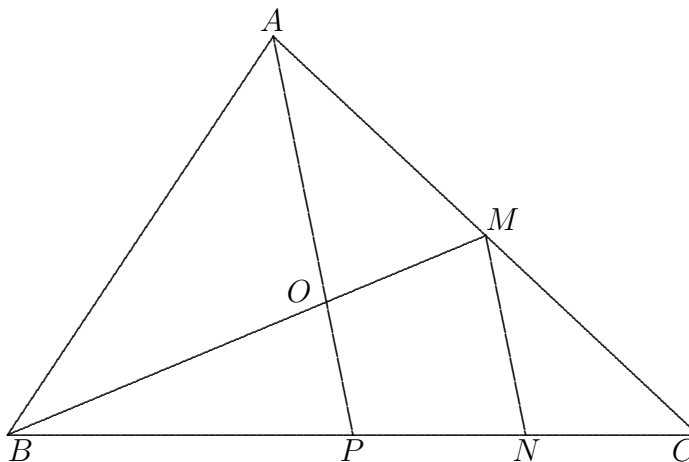
¹Courtesy of Andy Liu.

Solution to Junior A-Level Spring 2010

1. Solution by Olga Ivrii.

Let $\alpha > 1$. The first cut creates the piece $\frac{\alpha}{\alpha+1}$ and $\frac{1}{\alpha+1}$. Then cut the larger piece into $\frac{\alpha^2}{(\alpha+1)^2}$ and $\frac{\alpha}{(\alpha+1)^2}$. We want $\frac{\alpha^2}{(\alpha+1)^2} = \frac{\alpha}{(\alpha+1)^2} + \frac{1}{\alpha+1}$ or $\alpha^2 = \alpha + \alpha + 1$. Solving $\alpha^2 - 2\alpha - 1 = 0$, we have $\alpha = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$. Since $\alpha > 0$, $\alpha = \sqrt{2} + 1$.

2. Through M , draw a line parallel to AP , intersecting BC at N . Since triangles CMN and CAP are similar and $AM = MC$, $PN = NC$. Since triangles BOP and BMN are similar and $BO = BP$, $OM = PN$. Hence $\frac{OM}{PC} = \frac{OM}{2PN} = \frac{1}{2}$.



3. Since each product is equal to 1 or -1 , the value of S is always odd. Let the numbers be a_1, a_2, \dots, a_{999} in cyclic order, with $a_n = a_{n-999}$ whenever $n > 999$.

- (a) The minimum value of S is -997 . This can be attained by having 100 copies of -1 , each adjacent pair separated by 9 copies of 1, except for one pair which is separated by just 8 copies of 1. Every block of 10 adjacent numbers contains exactly 1 copy of -1 , the sole exception being the block with 1 copy of -1 at either end and 8 copies of 1 in between. If -997 is not the minimum value, it would have to be -999 , meaning that all 999 products are equal to -1 . Since $a_1 a_2 \cdots a_{10} = -1 = a_2 a_3 \cdots a_{11}$, we must have $a_1 = a_{11}$. Similarly, $a_{11} = a_{21} = \cdots = a_{9981}$. Since 10 and 999 are relatively prime, all these 999 subscripts are different. This means that we have either 999 copies of 1 or 999 copies of -1 . This is forbidden.
- (b) The maximum value of S is 995. This can be attained by having 2 adjacent copies of -1 and 997 copies of 1. There are only two blocks of 10 adjacent numbers which contain exactly one copy of -1 and have -1 as products. All other blocks have 1 as products. If 995 is not the maximum value, it would have to be 997 or 999. We cannot have 999 since this means all 999 numbers are copies of 1, or all are copies of -1 , which is forbidden. Suppose it is 997, which means that exactly one block of 10 adjacent numbers has product -1 . Let $a_1 a_2 \cdots a_{10} = -1$. Then $a_1 = -a_{11}$ but $a_{11} = a_{21} = \cdots = a_{9981}$. Since 10 and 999 are relatively prime, all these 999 subscripts are different. Hence all 999 numbers are equal except one, and there are exactly ten blocks of 10 adjacent numbers with product -1 , so that $S = 979$.

4. **Solution by Daniel Spivak.**

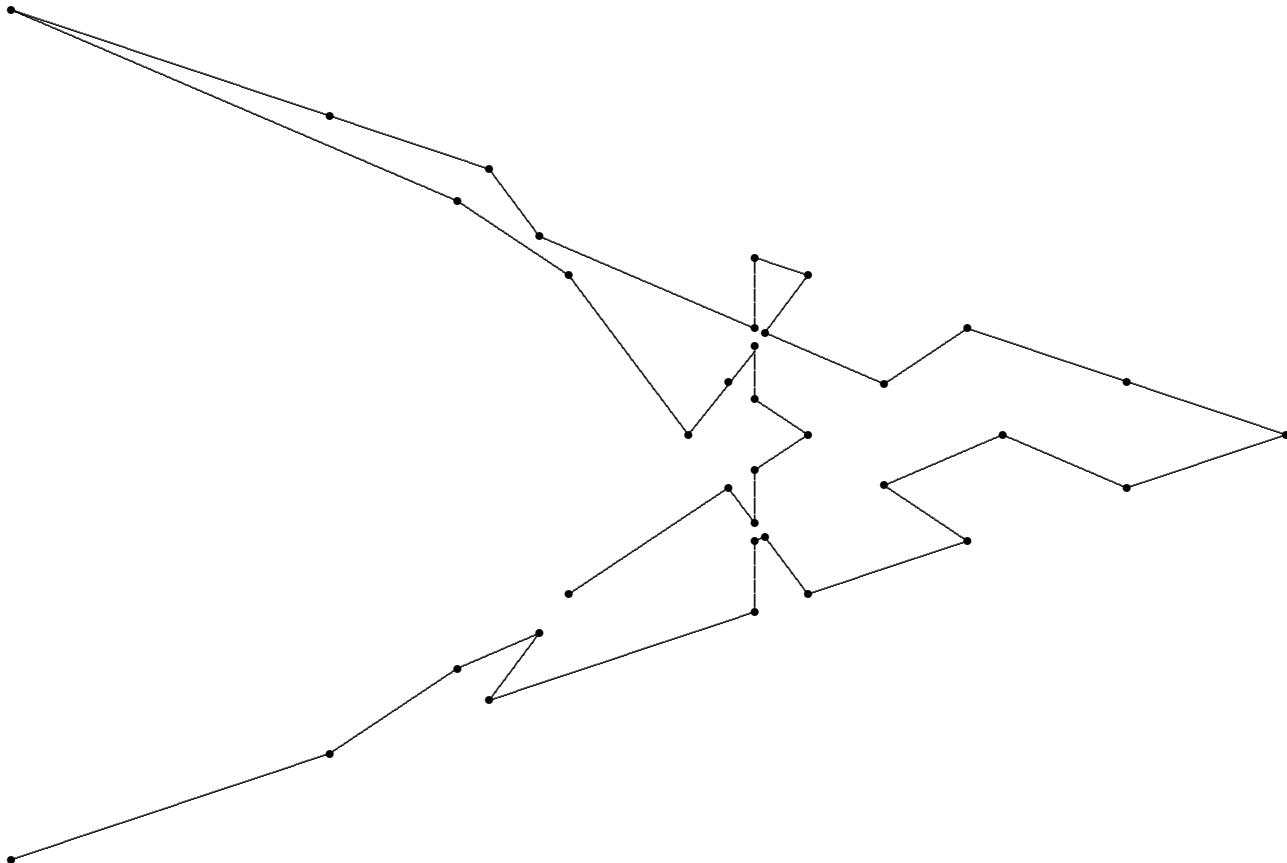
Let $n = 10^{4^1} + 10^{4^2} + \dots + 10^{4^{100}}$. Then the sum of the digits of n is 100. Consider n^3 . It is the sum of 100^3 terms each a product of three powers of 10. We claim that if two such terms are equal, they must be products of the same three powers of 10. If $4^a + 4^b + 4^c = 4^x + 4^y + 4^z$, where $a \leq b \leq c$ and $x \leq y \leq z \leq c$, we must have $z = c$. Otherwise, even if $x = y = z = c - 1$, we still have $3(4^{c-1}) < 4^c$. Similarly, we must have $y = b$ and $x = a$, justifying the claim. Now a product of the same three powers of 10 can occur at most $3! = 6$ times. Hence there is no carrying in adding these 100^3 terms, which means that the sum of the digits of n^3 is exactly 100^3 .

5. **Solution by Jonathan Zung.**

We use induction on the number n of runners. For $n = 1$, there is nothing to prove. Suppose the result holds for some $n \geq 1$, each with a distinct integer speed. Let M be the least common multiple of these speeds. If we add an $(n + 1)$ -st runner with speed 0 at the passing point, the result still holds. Now increase the speed of each of the $n + 1$ runners by M . Since their relative speeds remain the same, the result continues to hold. In particular, it holds for $n = 3$ and $n = 10$.

6. **Solution by Daniel Spivak.**

We first show that 9 lines are necessary. If we only have 8 lines, they generate at most 28 points of intersection. Since the broken line can only change directions at these points, it can have at most 29 segments. The diagram below shows a broken line with 32 segments all lying on 9 lines. Hence 9 lines are also sufficient.



7. **Solution by Peter Xie.**

We claim that on each row or column, there are at most 2 fleas, so that the total number of fleas on the board is at most $2 \times 10 + 2 \times 10 = 40$. Suppose there are 3 fleas on a row or column. By the Pigeonhole Principle, two of them must occupy cells of the same colour. These 2 fleas must occupy the same cell well before an hour has elapsed. This justified the claim. We now show a construction whereby there can be as many as 40 fleas on the chessboard.

	↓	→					←	↓	
↓	→			↓	↓			←	↓
→			↓			↓			←
	→	↓					↓	←	
	←							→	
	←							→	
	→	↑					↑	←	
→			↑			↑			←
↑	→			↑	↑			←	↑
	↑	→					←	↑	