International Mathematics

TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2010.**[1]

1. The exchange rate in a Funny-Money machine is s McLoonies for a Loonie or $\frac{1}{s}$ Loonies for a McLoonie, where s is a positive real number. The number of coins returned is rounded off to the nearest integer. If it is exactly in between two integers, then it is rounded up to the greater integer.
   (a) Is it possible to achieve a one-time gain by changing some Loonies into McLoonies and changing all the McLoonies back to Loonies?
   (b) Assuming that the answer to (a) is "yes", is it possible to achieve multiple gains by repeating this procedure, changing all the coins in hand and back again each time?

2. The diagonals of a convex quadrilateral $ABCD$ are perpendicular to each other and intersect at the point O. The sum of the inradii of triangles AOB and COD is equal to the sum of the inradii of triangles BOC and DOA.
   (a) Prove that $ABCD$ has an incircle.
   (b) Prove that $ABCD$ is symmetric about one of its diagonals.

3. From a police station situated on a straight road infinite in both directions, a thief has stolen a police car. Its maximal speed equals 90% of the maximal speed of a police cruiser. When the theft is discovered some time later, a policeman starts to pursue the thief on a cruiser. However, he does not know in which direction along the road the thief has gone, nor does he know how long ago the car has been stolen. Is it possible for the policeman to catch the thief?

4. A square board is dissected into n^2 rectangular cells by $n-1$ horizontal and $n-1$ vertical lines. The cells are painted alternately black and white in a chessboard pattern. One diagonal consists of n black cells which are squares. Prove that the total area of all black cells is not less than the total area of all white cells.

5. In a tournament with 55 participants, one match is played at a time, with the loser dropping out. In each match, the numbers of wins so far of the two participants differ by not more than 1. What is the maximal number of matches for the winner of the tournament?

Note: The problems are worth 2+3, 2+3, 5, 5 and 5 points respectively.

[1]Courtesy of Andy Liu

Solution to Senior O-Level Fall 2010

1. (a) For $s = 1$, there is obviously no chances of any gain. If $s > 1$ and we trade in n Loonies, we get $ns + \alpha$ McLoonies, where the real number α satisfies $0 \le \alpha \le \frac{1}{2}$. Then we trade these $ns + \alpha$ McLoonies back. The exchange rate is $\frac{ns+\alpha}{s} + \frac{\alpha}{s}$. Since $s > 1$, $\frac{\alpha}{s} < \frac{1}{2}$, so that there will not be any rounding up. Thus a one-time gain can only be possible if $s < 1$. Let $s = \frac{1}{2}$. We can trade in 1 Loonie for $\frac{1}{2}$ McLoonie, rounded up to 1. When we trade this back, we get 2 Loonies.

   (b) For an affirmative answer in (a), we must have $s < 1$. This means that $\frac{1}{s} > 1$, so that the argument in (a) shows that there will never be any increase in the number of McLoonies. So after the initial one-time gain in Loonies, no further gain is possible.

2. (a) The sum of the inradii of right triangles AOB and COD is given by

$$\frac{1}{2}(OA + OB - AB) + \frac{1}{2}(OC + OD - CD).$$

   The sum of the inradii of right triangles OCB and DOA is given by

$$\frac{1}{2}(OB + OC - BC) + \frac{1}{2}(OD + OA - DA).$$

   Hence the given condition is equivalent to $AB + CD = BC + DA$, which is the necessary and sufficient condition for $ABCD$ to have an incircle.

   (b) We may assume that among AB, BC, CD and DA, the longest one is DA. By Pythagoras' Theorem,

$$AB^2 + CD^2 = OA^2 + OB^2 + OC^2 + OD^2 = BC^2 + DA^2.$$

   Combined with $(AB + CD)^2 = (BC + DA)^2$, we have $AB \cdot CD = BC \cdot DA$. It follows that $(AB - CD)^2 = (BC - DA)^2$. If $DA - BC = AB - CD$, then $DA = AB$ and $ABCD$ is symmetric about AC. If $DA - BC = CD - AB$, then $DA = CD$ and $ABCD$ is symmetric about BD.

3. Let the speed of the cruiser be 1. The policeman's strategy is to go in one direction for a time period q, then go in the opposite direction for a time period q^2, and then go in the original direction for a time period q^3, and so on. At the end of the time period q^n, the total time elapsed is $q^n + q^{n-1} + \cdots + q = \frac{q^{n+1}-q}{q-1}$ and the net distance covered in the current direction is $q^n - q^{n-1} + \cdots + (-1)^n q = \frac{q^{n+1}-(-1)^n q}{q+1}$. Thus the net speed in this direction is $\frac{q^{n+1}-(-1)^n q}{q+1} \div \frac{q^{n+1}-q}{q-1} > \frac{q-1}{q+1}$. Solving for $\frac{q-1}{q+1} > \frac{9}{10}$, we have $q > 19$. Then the net speed of the cruiser still exceeds the raw speed of the car, and capture is inevitable.

4. Let the $n-1$ vertical lines divide the board into n vertical strips of respective width $a_1, a_2, \ldots,$ a_n, and let the $n-1$ horizontal lines divide the board into n horizontal strips of respective height $b_1, b_2, \ldots, b_n$. We may assume that the bottom left cell is a black square, and from the given conditions, we have $a_i = b_i$ for all $1 \le i \le n$. Consider the area of any black cell positive and the area of any white cell negative. Then the area of the cell of height a_i and width a_j is $(-1)^{i+j}a_ia_j$. The total area of the board is

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^{i+j}a_ia_j = \sum_{i=1}^{n}(-1)^i a_i \sum_{j=1}^{n}(-1)^j a_j = \left(\sum_{i=1}^{n}(-1)^i a_i\right)^2 \ge 0.$$

Hence the sum of the areas of the black cells cannot be less than the sum of the areas of the white cells.

5. Let a_n denote the total number of participants needed to produce a winner with n victories. We have $a_1 = 2$ and $a_2 = 3$. In order to have a winner with n victories, we must have a candidate with $n-1$ victories so far, and an also-ran with $n-2$ victories so far. Since no participant can lose to both of them, the total number of participants needed is $a_n = a_{n-1} + a_{n-2}$. Thus we have a shifted Fibonacci sequence. Iteration yields $a_3 = 5$, $a_4 = 8$, $a_5 = 13$, $a_6 = 21$, $a_7 = 34$ and $a_8 = 55$. Since we have 55 participants, the winner can have at most 8 victories. It is easy to construct recursively a tournament with 55 participants in which the winner has 8 victories.