International Mathematics

TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2010**[1]

1. In a multiplication table, the entry in the i-th row and the j-th column is the product ij. From an $m \times n$ subtable with both m and n odd, the interior $(m-2) \times (n-2)$ rectangle is removed, leaving behind a frame of width 1. The squares of the frame are painted alternately black and white. Prove that the sum of the numbers in the black squares is equal to the sum of the numbers in the white squares.

2. In a quadrilateral $ABCD$ with an incircle, $AB = CD$, $BC < AD$ and BC is parallel to AD. Prove that the bisector of $\angle C$ bisects the area of $ABCD$.

3. A $1 \times 1 \times 1$ cube is placed on an 8×8 chessboard so that its bottom face coincides with a square of the chessboard. The cube rolls over a bottom edge so that the adjacent face now lands on the chessboard. In this way, the cube rolls around the chessboard, landing on each square at least once. Is it possible that a particular face of the cube never lands on the chessboard?

4. In a school, more than 90% of the students know both English and German, and more than 90% of the students know both English and French. Prove that more than 90% of the students who know both German and French also know English.

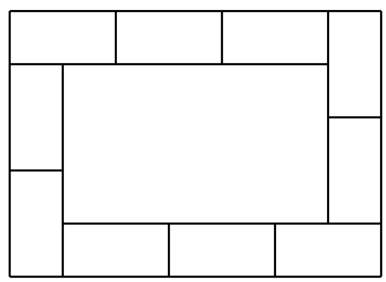
5. A circle is divided by $2N$ points into $2N$ arcs of length 1. These points are joined in pairs to form N chords. Each chord divides the circle into two arcs, the length of each being an even integer. Prove that N is even.

Note: The problems are worth 4, 4, 4, 4 and 4 points respectively.

[1]Courtesy of Andy Liu

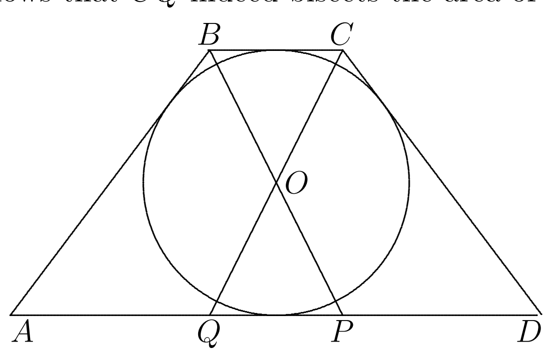
Solution to Junior O-Level Fall 2010

1. **First Solution.** Let the top row of the frame be row a, the bottom row be row b, the left column be column c and the right column be column d. From the given condition, the frame can be partitioned into dominoes as shown in the diagram below. We may assume that one of the corner squares is black. Then all corner squares are black. Consider all numbers in black squares positive and all numbers in white squares negative. In row a, there are $\frac{d-c}{2}$ dominoes each with sum $-a$. In row b, there are $\frac{d-c}{2}$ dominoes each with sum b. In column c, there are $\frac{b-a}{2}$ dominoes each with sum c, and in column d, there are $\frac{b-a}{2}$ dominoes each with sum $-d$. Hence the grand total is $\frac{1}{2}(-a(d-c)+b(d-c)+c(b-a)-d(b-a))=0$, meaning that the sum of all numbers in black squares is equal to the sum of all numbers in white squares.



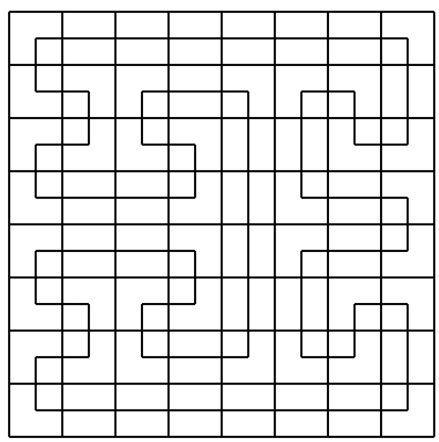
Second Solution.
Consider the number c at the centre of the original $m \times n$ rectangle. The two numbers on the frame in the central column is symmetric about c. They are of the same colour, and their average is just c. The same is true of the two numbers on the frame in the central row. The other numbers on the frame may be divided into sets of four, each set defining a rectangle with sides parallel to the sides of the table. The four numbers in each set are of the same colour, and their average is also c. Since we have an equal number of black squares and white squares, the total of the numbers on the black squares must be equal to the total of the numbers on the white squares.

2. **Solution by Weilian Chu.**
The bisectors of $\angle B$ and $\angle C$ both pass through the centre O of the circle. Let them intersect AD at P and Q respectively. By symmetry, $OCDP$ and $OBAQ$ are congruent. Triangles OBC and OPQ are also congruent as they are isosceles triangles with equal vertical angle and equal altitude. It follows that CQ indeed bisects the area of $ABCD$



3. **First Solution by Janet Leahy.**
Place the die on square a1 with face 1 in front. Roll the die through to h1, then back to g1, onto g2 and over to h2. During this motion, face 1 never touches the chessboard, and is now on the right. Roll the die through to h8, then back to h3, onto g3 and back to g2. During this motion, face 1 never touches the chessboard, and is now in front again. Roll the die through to a2, back to f2, onto f3 and over to g3. During this motion, face 1 never touches the chessboard, and is now on the right again. Continuing this way, it can cover each square at least once, without face 1 ever touching the chessboard.

Second Solution.
It is possible, and the path is indicated in the diagram below. For convenience in description, we use a cubical die starting on the top left corner with the face 2 at the bottom, the face 3 to the left, the face 5 on top, the face 4 to the right, the face 6 to the front and the face 1 to the back. The faces that land on the chessboard are (4,5,3,2,4,5,3), (6,4), (5), (6), (3), (5,4), (6,3), (2), (6,5), (4,2), (6), (4), (5), (6,2), (4,5,3,2,4,5,3), (6), (5), (4), (6), (2), (4,5,3), (6), (5), (4), (6,3), (5,4,2,3,5), (6,2), (3), (6), (5), (3,2,4), (6), (2), (3), (6) and (5). The face 1 never lands on the chessboard.



4. Let the total number of students be T, the number of those who know English, French and German w, the number of those who know English and French but not German x, the number of those who know English and German but not French y, and the number of those who know French and German but not English z. We are given that $\frac{w+x}{T} > \frac{9}{10}$ and $\frac{w+y}{T} > \frac{9}{10}$. From $\frac{w+x}{w+x+y+z} \geq \frac{w+x}{T} > \frac{9}{10}$, we have $w + y > 9(x + z)$. Simlarly, we have $w + x > 9(y + z)$. Hence $2w + 9(x + y) > (w + x) + (w + y) > 9(x + z) + 9(y + z) = 9(x + y) + 18z$. It follows that $w > 9z$ and $10w > 9(w + z)$, so that $\frac{w}{w+z} > \frac{9}{10}$.

5. Paint the $2N$ endpoints of the N chords alternately yellow and blue around the circle. Since a chord divides the circle into two arcs of even length, its two endpoints must have the same colour. Since there are N endpoints of each colour, N must be even.