International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2010.**[1]

1. A round coin may be used to construct a circle passing through one or two given points on the plane. Given a line on the plane, show how to use this coin to construct two points such that they define a line perpendicular to the given line. Note that the coin may not be used to construct a circle tangent to the given line.

2. Pete has an instrument which can locate the midpoint of a line segment, and also the point which divides the line segment into two segments whose lengths are in a ratio of $n : (n+1)$, where n is any positive integer. Pete claims that with this instrument, he can locate the point which divides a line segment into two segments whose lengths are at any given rational ratio. Is Pete right?

3. At a circular track, 10 cyclists started from some point at the same time in the same direction with different constant speeds. If any two cyclists are at some point at the same time again, we say that they meet. No three or more of them have met at the same time. Prove that by the time every two cyclists have met at least once, each cyclist has had at least 25 meetings.

4. A rectangle is divided into 2×1 and 1×2 dominoes. In each domino, a diagonal is drawn, and no two diagonals have common endpoints. Prove that exactly two corners of the rectangle are endpoints of these diagonals.

5. For each side of a given pentagon, divide its length by the total length of all other sides. Prove that the sum of all the fractions obtained is less than 2.

6. In acute triangle ABC, an arbitrary point P is chosen on altitude AH. Points E and F are the midpoints of sides CA and AB respectively. The perpendiculars from E to CP and from F to BP meet at point K. Prove that $KB = KC$.

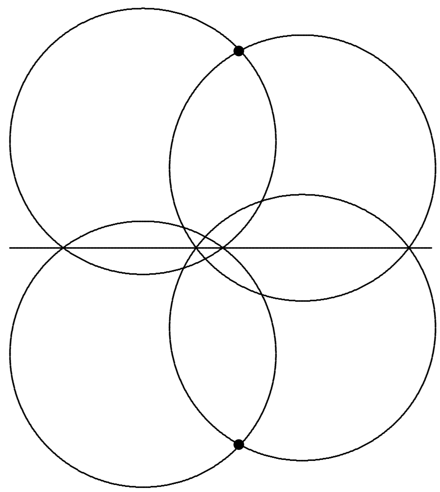
7. Merlin summons the n knights of Camelot for a conference. Each day, he assigns them to the n seats at the Round Table. From the second day on, any two neighbours may interchange their seats if they were not neighbours on the first day. The knights try to sit in some cyclic order which has already occurred before on an earlier day. If they succeed, then the conference comes to an end when the day is over. What is the maximum number of days for which Merlin can guarantee that the conference will last?

Note: The problems are worth 4, 5, 8, 8, 8, 8 and 12 points respectively.

[1]Courtesy of Andy Liu

Solution to Junior A-Level Fall 2010

1. Take two points on the given line at a distance less than the diameter of the round coin. Draw two circles passing through these two points. They are situated symmetrically about the given line. Repeat this operation so that a circle from the second operation intersects a circle from the first operation. The point symmetric to this point about the given line is also a point of intersection of two constructed circles, one from each operation. These two points satisfy the requirement of the problem.

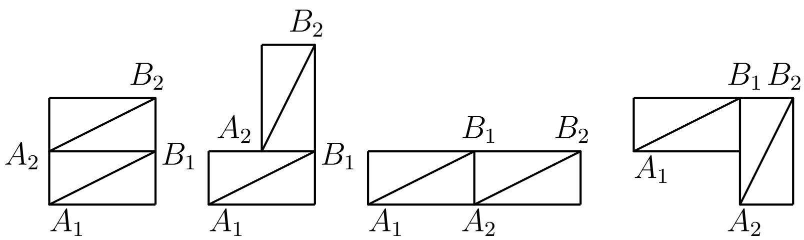


2. Pete is right. Suppose he is asked to divide a line segment into two whose lengths are in the ratio $p : q$, where p and q are relatively prime integers. We may assume that the line segment has length $p+q$. Then Pete can in fact divide the line segment into $p+q$ unit segments. For any segment of length greater than 1, if it has even length, Pete divides it by its midpoint. If it has odd length $2n+1$, Pete divides it into two whose length are in the ratio $n : (n+1)$. Eventually, all segments are of length 1. It is then easy to pick out the point which divides the segment into two whose lengths are in the ratio $p : q$.

3. For $1 \leq i \leq 10$, let the constant speed of cyclist C_i be v_i, where $v_1 < v_2 < \cdots < v_{10}$. Let $u = \min\{v_2 - v_1, v_3 - v_2, \ldots, v_{10} - v_9\}$. Then $v_j - v_i \geq (j-i)u$ for all $j > i$. Let d be the length of the track. Then the meeting between the last pair of cyclists occurs at time $\frac{d}{u}$. Now C_i and C_j meet once in each time interval of length $\frac{d}{v_j - v_i}$. They would have met at least $j-i$ times by the time of the meeting of the last pair, because $(j-i)\frac{d}{v_j - v_i} \leq \frac{d}{u}$. For C_i, this means at least $1+2+\cdots+(i-1)$ meetings with $C_1,\ C_2,\ \ldots,\ C_{i-1}$ and at least $1+2+\cdots+(10-i)$ meetings with $C_{i+1},\ C_{i+2},\ \ldots,\ C_{10}$. The total is at least

$$\frac{i(i-1)+(10-i)(10-i+1)}{2} = (i-5)(i-6)+25 \geq 25.$$

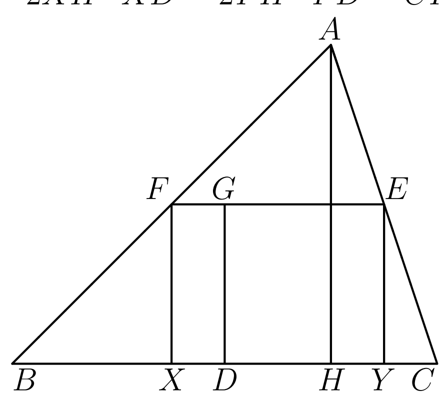
4. **Solution by Central Jury.**
We first prove that at least one corner of the rectangle is the endpoint of a diagonal. Consider the domino at the bottom right corner. If its diagonal is in the down direction (from the left), then the bottom right corner is the endpoint of a diagonal. Suppose its diagonal is in the up direction. Consider all the dominoes touching the bottom edge of the rectangle. All of their diagonals must be in the up direction, which means that the bottom left corner of the rectangle is the endpoint of a diagonal. Henceforth, we assume that the bottom left corner is the endpoint A_1 of an up-diagonal A_1B_1 of a domino. The next domino has B_1 as one of its vertices but not the one at the bottom left corner. The diagram below illustrates some of the possible choices. The diagonal A_2B_2 in this domino must also be in the up direction. Note that B_2 is above B_1, to the right of B_1 or both.



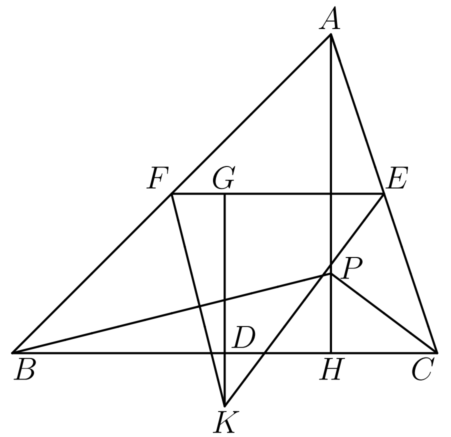
Continuing in this manner, we can build a connected chain of dominoes with up-diagonals going from the bottom left vertex to the upper right vertex of the rectangle. It is impossible to build simultaneously another connected chain of dominoes with down-diagonals going from the bottom right vertex to the upper left vertex of the rectangle. This is because two such chains must share a common domino, and only one diagonal of that domino is drawn.

5. Let the side lengths be $a_1 \le a_2 \le \cdots \le a_5 < p - a_5$ where $p = a_1 + a_2 + \cdots + a_5$. We have $\frac{a_1}{p-a_1} + \frac{a_2}{p-a_2} + \cdots + \frac{a_5}{p-a_5} \le \frac{a_1}{p-a_5} + \frac{a_2}{p-a_5} + \cdots + \frac{a_5}{p-a_5} = \frac{p}{p-a_5} < 2$ since $p > 2a_5$.

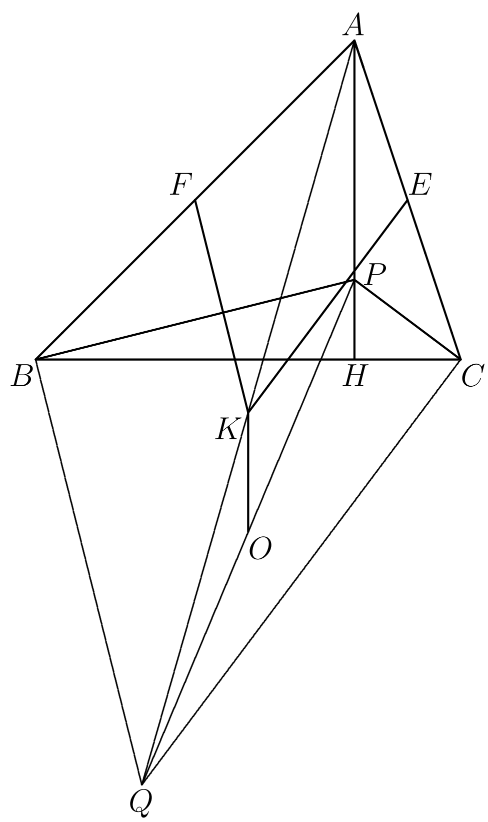
6. **First Solution:**
We first prove an auxiliary result. Let the line through the midpoint D of BC and perpendicular to BC cut EF at G. Then $BH \cdot FG = CH \cdot EG$. Drop perpendiculars FX and EY from F and E to BC respectively. Then X is the midpoint of BH and Y is the midpoint of CD. Since triangles FXD and EYC are congruent, we have $XD = YC = YH$ so that $XH = YD$. Now $BH \cdot FG = 2XH \cdot XD = 2YH \cdot YD = CH \cdot EG$.



Returning to the main problem, let the line through F perpendicular to BP intersect the line DG at K_1. Triangles FGK_1 and PHB are similar. Hence $GK_1 = \frac{FG \cdot BH}{PH}$. Let the line through E perpendicular to CP intersect the line DG at K_2. We can prove in an analogous manner that $GK_2 = \frac{EG \cdot CH}{PH}$. By the auxiliary result, $GK_1 = GK_2$. Hence K_1 and K_2 is the same point K. Since K lies on the perpendicular bisector of BC, we have $KB = KC$.



Second Solution by Chun-Yu Yang.
Extend AK to Q so that K is the midpoint of AQ. Since F is the midpoint of AB, BQ is parallel to FK, which is perpendicular to BP. Hence $\angle PBQ = 90°$. Similarly, $\angle PCQ = 90°$. Hence the midpoint O of PQ is the circumcentre of the cyclic quadrilateral $BPCQ$. Now OK is parallel to PA, which is perpendicular to BC. Hence K lies on the perpendicular bisector of BC, so that $KB = KC$.



7. **Solution by Central Jury.**

We may assume that the Knights are seated from 1 to n in cyclic clockwise order on day 1. Then seat exchanges are not permitted between Knights with consecutive numbers (1 and n are considered consecutive). We construct an invariant for a cyclic order called the winding number as follows. Merlin has n hats numbered from 1 to n from top to bottom. He starts by giving hat 1 to Knight 1. Then he continues in the clockwise order round the table until he gets to Knight 2, when he will give him hat 2. After he has handed out all the hats, Merlin returns to Knight 1 which is his starting point. The number of times he has gone round the table is called the winding number of the cyclic order. For instance, the winding number of the cyclic order 1 4 7 2 3 6 5 is 4: (1,2,3)(4,5)(6)(7). Suppose two adjacent Knights change seats. If neither is Knight 1, the hats handed out in each round remain the same, so that the winding number remains constant. Suppose the seat exchange is between Knight 1 and Knight h, where $h \neq 2$ or n. Then Knight h either becomes the first Knight to get a hat in the next cycle instead of the last Knight to get a hat in the preceding cycle, or vice versa. Still, the winding number remains constant. On the k-th day of the conference, $1 \leq k \leq n-1$, Merlin can start by having the Knights sit in the cyclic order $k, k-1, \ldots, 2, 1, k+1, k+2, \ldots, n$. It is easy to verify that the winding number of the starting cyclic order on the k-th day is k. It follows that no cyclic order can repeat on two different days within the first $n-1$ days. Therefore, Merlin can make the conference last at least n days. We claim that given any cyclic order, it can be transformed into one of those with which Merlin starts a day. Then the Knights can make the conference end when the n-th day is over. Suppose the cyclic order is not one with which Merlin starts a day. We will push Knight 2 forward in clockwise direction until he is adjacent to Knight 1. This can be accomplished by a sequence of exchanges if he does not encounter Knight 3 along the way. If he does, we will push both of them forward towards Knight 1. Eventually, we will have Knights 2, 3, $\ldots$, h and 1 in a block. Now $h < n$ as otherwise the initial cyclic order is indeed one of those with which Merlin starts a day. Hence we can push Knight 1 counter-clockwise so that he is adjacent to Knight 2. We now attempt to put Knight 3 on the other side of Knight 2. As before, we have Knights 3, 4, $\ldots$, ℓ, 2, 1 in a block. If $\ell < n$, we can push Knights 2 and 1 counter-clockwise towards Knight 3. If ℓ, Knight 1 cannot get past, but we notice that we have arrived at one of the cyclic orders with which Merlin starts a day. This justifies the claim.