International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Spring 2009.**

1. Let $a \wedge b$ denote the number a^b. The order of operations in the expresson $7 \wedge 7 \wedge 7 \wedge 7 \wedge 7 \wedge 7 \wedge 7$ must be determined by inserting five pairs of brackets. Is it possible to put brackets in two distinct ways so that the expressions have the same value?

2. Several points on the plane are such that no three lie on a straight line. Some pairs of points are connected by segments. If any line which does not pass through any of these points intersects an even number of these segments, prove that each of these points is connected to an even number of the other points.

3. Let a and b be arbitrary positive integers. The sequence $\{x_k\}$ is defined by $x_1 = a$, $x_2 = b$ and for $k \geq 3$, x_k is the greatest common divisor of $x_{k-1} + x_{k-2}$.
   (a) Prove that the sequence is eventually constant.
   (b) How can this constant value be determined from a and b?

4. In an arbitrary binary number, consider any digit 1 and any digit 0 which follows it, not necessarily immediately. They form an odd pair if the number of other digits in between is odd, and an even pair if this number is even. Prove that the number of even pairs is greater than or equal to the number of odd pairs.

5. X is an arbitrary point inside a tetrahedron. Through each of the vertices of the tetrahedron, draw a line parallel to the line joining X to the centroid of the opposite face. Prove that these four lines are concurrent.

Note: The problems are worth 3, 4, 2+2, 4 and 4 points respectively.

Courtesy of Andy Liu

Solution to Senior O-Level Spring 2009

1. **Solution by Olga Ivrii:**
   More generally, $(n \wedge (n \wedge n)) \wedge n = (n^{n \wedge n})^n = (n^n)^{n \wedge n} = (n \wedge n) \wedge (n \wedge n)$. Adding three more terms to both sides the same way maintains the equal value.

2. Suppose there is a point A connected to an odd number of other points. Then there must be a second such point B, because each connection involves two points. Take a line very close to AB, so that it does not pass through any given point. This line cuts a segments connnect to A, b segments connected to B and c segments not connected to A or B, where $a+b+c$ is an even number. We now rotate this line slightly so that A remains on the same side but B moves to the opposite side of this line. Apart from possibly AB, this line cuts a segments connected to A, d segments connected to B and c segments not connected to A or B. If A is connected to B, then $d-b$ is even, and the total count $1+a+d+c = 1+a+b+c+(d-b)$ is odd. If A is not connected to B, then $d-b$ is odd, and the total count $a+d+c = a+b+c+(d-b)$ is still odd. In either case, we have a contradiction.

3. **Solution by Olga Ivrii:**
   (a) Note that x_k is odd for all $k \geq 3$. Hence $x_{k+2} = \frac{x_{k+1}+x_k}{2^t}$ for some positive integer t. If $x_{k+1} = x_k$, then $t = 1$ and the sequence in constant from this point on. Otherwise, $x_{k+2} < \max\{x_{k+1}, x_k\}$. Similarly, $x_{k+3} < \max\{x_{k+2}, x_{k+1}\}$. If $x_k < x_{k+1}$, then $x_{k+2} < x_{k+1}$ and $x_{k+3} < x_{k+1}$. If $x_k > x_{k+1}$, then $x_{k+2} < x_k$ and $x_{k+3} < x_k$. Thus $\max x_{k+3}, x_{k+2}\}, \max\{x_{k+1}, x_k$, so that the sequence is essentially decreasing, though not monotonically. Since the terms are positive integers, an infinite descent is impossible. Hence the sequence must eventually be constant.

   (b) Let g be the greatest common odd divisor of a and b. Then g is an odd divisor of x_k for $k \geq 3$. Hence it is the greatest common odd divisor of x_{k+1} and x_k. When the sequence becomes constant, g is the greatest common odd divisor of two equal terms both of which are odd. Hence this constant term must be equal to g.

4. We claim that if there is a pair of adjacent 0s, then they may be removed. This affects two kinds of pairs, those in which the 0 is one of the digits removed, and those in which the 0 comes after the digits removed. Among the first kind, whenever one of the removed 0 had formed part of an odd pair, the other removed 0 had formed part of an even pair with the same digit 1, and vice versa. Among the second kind, odd pairs remain odd and even pairs remain even with the removal of the two 0s. This justifies our claim. Similarly, pairs of adjacent 1s may be removed. When no pairs of adjacent digits are identical, the digits of the number are alternately 0 and 1. If we are left with a 0-digit or 1-digit number, then the numbers of odd and even pairs are both 0. Suppose we are left with a number with at least 2 digits. Since leading 0s and trailing 1s do not count, we may assume that our number has the form $1010\cdots10$. Clearly all pairs are even.

5. Let O be the centroid of the tetrahedron $ABCD$ and G be the centroid of the face BCD. Then O lies on AG, with $AO = 3OG$. Let P be the point on the extension of XO such that $PO = 3OX$. Then triangles GOX and AOP are similar, so that XG is parallel to AP. By symmetry, the fixed point P lies on each of the four lines.