International Mathematics
# TOURNAMENT OF THE TOWNS

**Senior A-Level Paper** **Spring 2009.**

1. A rectangle is dissected into several smaller rectangles. Is it possible that for each pair of rectangles so obtained, the line segment joining their centres intersects some other rectangle?

2. In a sequence of distinct positive integers, each term except the first is either the arithmetic mean or the geometric mean of the term immediately before and the term immediately after. Is it necessarily true that from a certain point on, the means are either all arithmetic means or all geometric means?

3. There is a counter is each square of a $10 \times 10$ board. We may choose a diagonal containing an even number of countes and remove any counter from it. What is the maximum number of counters which can be removed from the board by these operations?

4. Three planes dissect a parallelepiped into eight hexahedra such that all of their faces are quadrilaterals. One of the hexahedra has a circumsphere. Prove that each of these hexahedra has a circumsphere.

5. Let $\binom{n}{k}$ be the number of ways of choosing a subset of $k$ objects from a set of $n$ objects. Prove that if $k$ and $\ell$ are positive integers less than $n$, then $\binom{n}{k}$ and $\binom{n}{\ell}$ have a common divisor greater than 1.

6. A positive integer $n$ is given. Two players take turns marking points on a circle. The first player uses the red colour while the second player uses the blue colour. When $n$ points of each colour has been marked, the game is over,and the circle has been divided into $2n$ arcs. The winner is the player who has the longest arc both endpoints of which are of this player's colour. Which player can always win, regardless of any action of the opponent?

7. At step 1, the computer has the number 6 in a memory cell. In step $n$, it computes the greatest common divisor $d$ of $n$ and the number $m$ currently in that cell, and replaces $m$ with $m + d$. Prove that if $d > 1$, then $d$ must be prime.

**Note:** The problems are worth 4, 4, 6, 6, 8, 9 and 9 points respectively.

Courtesy of Andy Liu

## Solution to Senior A-Level Spring 2009

1. Place the original rectangle in the first quadrant so that its southwest corner coincides with the origin of the coordinate plane. The small rectangle whose southwest corner also coincides with the origin is called the main rectangle. Let its centre be at $(x, y)$. Consider the rectangles which have parts of the northern edge of the main rectangle as their southern edges. Let there be $n$ of them and let their centres be at $(x_i, y_i)$, with $x_1 < x_2 < \cdots < x_n$. Let $k$ be such that $x_k < x < x_{k+1}$. The segment joining $(x_k, y_k)$ and $(x, y)$ must therefore pass through the $(k+1)$-st rectangle, which means that the segment joining $(x, y)$ and $(x_{k+1}, y_{k+1})$ cannot pass through the $k$-th rectangle. It follows that it must intersect the eastern edge of the main rectangle rather than its northern edge, so that $n = k+1$. Similarly, if there are $m$ rectangles which have parts of the eastern edge of the main rectangle as their western edges, the line segment joining $(x, y)$ to the centre of the last of these rectangles must intersect the northern edge of the main rectangle rather than its eastern edge. However, the $n$-th northern neighbour and the $m$-th eastern neighbour of the main rectangle share common interior points, which is a contradiction. Thus it is not possible that for each pair of rectangles so obtained, the line segment joining their centres intersects some other rectangle.

2. **Solution by Daniel Spivak and Yu Wu, independently:**
   In the sequence defined by $a_{2k-1} = k^2$ and $a_{2k} = k(k+1)$ for all $k \geq 1$, we have

   $$\sqrt{a_{2k-1}a_{2k+1}} = \sqrt{k^2(k+1)^2} = k(k+1) = a_{2k}$$

   while

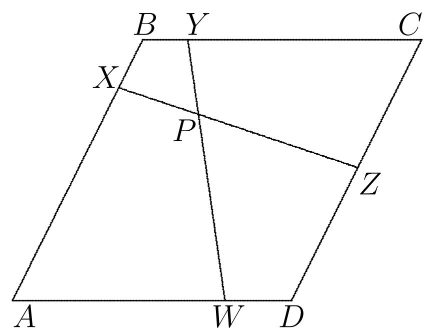
   $$\frac{1}{2}(a_{2k} + a_{2k+2}) = \frac{1}{2}(k(k+1) + (k+1)(k+2)) = (k+1)^2 = a_{2k+1}.$$

   Hence the means are alternately geometric and arithmetic.

3. **Solution by Zimu Zhu:**
   The status of a diagonal is the parity of the number of counters currently on it. Initially, twenty of them are odd. Whenever a counter is removed, it affects the status of the two diagonals on which it lies. They cannot both be odd. If one is odd and the other is even, the total number of odd diagonals remains the same. If both are even, that number increases by two. Hence it cannot fall below its initial value of twenty. It follows that at least ten counters must remain on the board. Label the squares $(i, j)$ where $0 \leq i, j \leq 9$. We can remove all but five counters on the squares $(i, j)$ where $i + j$ is odd, namely (1,0), (3,0), (5,0), (7,0) and (9,0). This is accomplished in ten stages by removing the counters on the squares, using even diagonals in alternating directions:
   0. (0,1), (0,3), (0,5), (0,7) and (0,9),
   1. (1,2), (1,4), (1,6), (1,8).
   2. (2,1), (2,3), (2,5), (2,7) and (2,9),
   3. (3,2), (3,4), (3,6), (3,8).
   4. (4,1), (4,3), (4,5), (4,7) and (4,9),
   5. (5,2), (5,4), (5,6), (5,8).
   6. (6,1), (6,3), (6,5), (6,7) and (6,9),
   7. (7,2), (7,4), (7,6), (7,8).
   8. (8,1), (8,3), (8,5), (8,7) and (8,9),
   9. (9,2), (9,4), (9,6), (9,8).

   Similarly, we can remove all but five counters on the squares $(i, j)$ where $i + j$ is even, namely (0,0), (2,0), (4,0), (6,0) and (8,0). Thus only the ten counters on the first row are left.

4. Our solution makes use of the following two auxillary results.
**Lemma 1.** Let $ABCD$ be a parallelogram. Let a line intersect $AB$ at $X$ and $CD$ at $Z$. Let another line intersect $BC$ at $Y$ and $DA$ at $W$. Let $WY$ intersect $XZ$ at $P$. If the quadrilateral $AXPW$ is cyclic, then so are the quadrilaterals $BYPX$, $CZPY$ and $DWPZ$.
**Proof:**
Let $\angle WAX = \theta$. Then $\angle YCZ = \theta$ also since $ABCD$ is a parallelogram. We also have $\angle WPZ = \angle XPY = \theta$ since $AXPW$ is a cyclic quadrilateral. It follows that so is $CZPY$. Now $\angle XBY = \angle ZDW = \angle WPX = \angle YPZ = 180° - \theta$. Hence $BYPX$ and $DWPZ$ are cyclic quadrilaterals also.




**Lemma 2.** Let $AXPW$ be a face of a hexahedron. Let $A'X'P'W'$ be the opposite face such that $AA'$, $XX'$, $PP'$ and $WW'$ are the lateral edges. If all six faces are cyclic quadrilaterals, then the hexahedron itself is cyclic.
**Proof:**
Since $AXPW$ is cyclic, there are many spheres which contains its circumcircle, and we can find one which passes through $A'$. Now $A$, $X$ and $A'$ determines a unique circle. It must be the circumcircle of $AXA'X'$, and it must lie on this sphere. It follows that $X'$ lies on this sphere. The same argument shows that $P'$ and $W'$ also lie on this sphere, so that the hexahedron is cyclic.
We now return to the original problem. Each of the three planes has a cross-section with the parallelepiped in the form of a parallelogram. This cross-section does not meet two opposite faces of the parallelepiped, which are also parallelograms. All three parallelograms are divided into four quadrilaterals. In two of these parallolgrams, one of the four quadrilaterals is cyclic. By Lemma 1, the others are also cyclic. In the third parallelogram, which is a face of the parallelepiped, the dividing lines form the same angles with the sides of the parallelogram as those in the opposite face. Hence the four quadrilaterals here are cyclic too. It follows that all faces of the eight hexahedra are cyclic. By Lemma 2, the hexahedra are all cyclic.

5. **Solution by Jonathan Schneider:**
Let $0 < k < \ell < n$. Then $\binom{\ell}{k} < \binom{n}{k}$. Suppose we have $n$ players from which we wish to choose a team of size $\ell$, and to choose $k$ captains among the team players. The team can be chosen in $\binom{n}{\ell}$ ways and the captains can be chosen in $\binom{\ell}{k}$ ways. On the other hand, if we choose the captains first among all the players, the number of ways is $\binom{n}{k}$. From the remaining $n-k$ players, there are $\binom{n-k}{\ell-k}$ ways of choosing the $\ell-k$ non-captain players. Hence $\binom{n}{\ell}\binom{\ell}{k} = \binom{n}{k}\binom{n-k}{\ell-k}$. Now $\binom{n}{k}$ divides $\binom{n}{\ell}\binom{\ell}{k}$. If it is relatively prime to $\binom{n}{\ell}$, then it must divide $\binom{\ell}{k}$. This is a contradiction since $\binom{\ell}{k} < \binom{n}{k}$.

6. **Solution by Jonathan Zung:**
The second player has a winning strategy divided into four stages.

   1. After the first player has marked the initial red point, the second player define as principal points the remaining vertices of a regular $n$-gon inscribed in the circle, with this red point as one of the vertices.
   2. The second player marks principal points whenever possible, until all have been marked. Since the second player has $n$ moves and there are only $n-1$ unmarked principal points initially, this stage ends before the second player's last move.
   3. Once all the principal points have been marked, the second player find pairs of adjacent red principal points. For each such pair, the second player marks a blue point between the two red points. Suppose the first player has marked $k$ principal points red while the second player has marked the remaining $n-k$ principal points blue. There are at most $k-1$ pairs of adjacent red principal points. Hence this stage also ends before the second player's last move.
   4. When the second player is ready to make the last move, all $n$ principal points have been marked. There are $n-1$ other marked points. Hence there exist two adjacent principal points with no other points in between. At least one of them is blue since the second player ensured there is a blue point between any two adjacent red principal points. The second player's final marked point is on this arc, arbitrarily close to a principal point where the other principal is blue.

   The longest arc the second player can claim may be made arbitrarily close to $\frac{1}{n}$ of the circle, while all arcs the first player can claim are shorter than $\frac{1}{n}$ of the circle. Hence the second player can be assured of a win regardless of any action by the first player.

7.

See on the next page

**7** Let us write down few first terms in the sequence:

| Step # | | 1 | 2 | 3 | 4 | **5** | **6** | 7 | 8 | 9 | 10 | **11** | **12** | … |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Number in the cell | 6 | 7 | 8 | 9 | 10 | **15** | **18** | 19 | 20 | 21 | 22 | **33** | **36** | … |
| Increment | | 1 | 1 | 1 | 1 | **5** | **3** | 1 | 1 | 1 | 1 | **11** | **3** | |

Let us denote by $n$ the number of the step, $A(n)$ the number in the cell, $I(n) = A(n) - A(n-1)$ its increment.

One can notice the following pattern: *If on some step* $n$ $I(n) \neq 1$ *then* $A(n) = 3n$. (In the table corresponding columns are in bold).

Let $A(n) = 3n$ for some $n$. On the next step the number increases by $I(n+1) = \gcd(n+1, 3n)$ and since $n$ and $(n+1)$ are coprimes then $I(n+1) = \gcd(n+1, 3)$. Thus, increment is either $I(n+1) = 1$ or $I(n+1) = 3$. In the latter case we have that $(n+1)$ is divisible by 3 so on the next step $I(n+2) = 1$ for certain.

This observation leads us to the following

**Conjecture** *Let* $A(n) = 3n$ *for some* $n$*, and the next increment be* $I(n+1) = 1$*. Consider the nearest step* $n+k$ *when increment will be different from 1:* $I(n+k) \neq 1$*. Then* $I(n+k)$ *is a prime number and* $A(n+k) = 3(n+k)$*.*

To prove conjecture we use induction. We already checked the base for small numbers $n$. Let $A(n) = 3n$ for some $n$ and $(n+k)$ be the nearest number with $I(n+k) \neq 1$:

| Step # | n | n+1 | n+2 | … | n+k-1 | n+k |
|---|---|---|---|---|---|---|
| Number in the cell | 3n | 3n+1 | 3n+2 | … | 3n+k-1 | ? |

For increment $I(n+k)$ we have (using here and below $\gcd(a, b) = \gcd(a, a-b)$):

$$I(n+k) = \gcd(n+k, 3n+k-1) = \gcd\big(n+k, 3(n+k) - (3n+k-1)\big) = \gcd(n+k, 2k+1).$$

Hence, $I(n+k)$ is divisor of $(2k+1)$.

Assume that $(2k+1)$ is not a prime and $p$ is a prime divisor of $\gcd(n+k, 2k+1)$. Since $(2k+1)$ is odd then $p \leq (2k+1)/3$. Therefore $p < k$. Let us look at step $n+k-p$. At this step an increment is

$$I(n+k-p) = \gcd(n+k-p, 3n+k-p-1) = \gcd\big(n+k-p, 3(n+k-p) - (3n+k-p-1)\big) = \gcd(n+k-p, 2k+1-2p).$$

But since both $(n+k-p)$ and $(2k+1-2p)$ are divisible by $p$ we see that on step $(n+k-p)$ increment differs from 1. This contradicts to the assumption that $(n+k)$ is the nearest step.

Therefore, $(2k+1)$ is a prime number and $I(n+k) = 2k+1$ and then

$$A(n+k) = A(n+k-1) + I(n+k) = (3n+k-1) + (2k+1) = 3(n+k).$$

Our conjecture is proven by induction and the problem solved.