International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2009.**[1]

1 [**4**] A rectangle is dissected into several smaller rectangles. Is it possible that for each pair of these rectangles, the line segment connecting their centers intersects some third rectangle?

2 [**4**] Consider an infinite sequence consisting of distinct positive integers such that each term (except the first one) is either an arithmetic mean or a geometric mean of two neighboring terms. Does it necessarily imply that starting at some point the sequence becomes either arithmetic progression or a geometric progression?

3 [**6**] Each square of a 10×10 board contains a chip. One may choose a diagonal containing an even number of chips and remove any chip from it. Find the maximal number of chips that can be removed from the board by these operations.

4 [**6**] Three planes dissect a parallelepiped into eight hexahedrons such that all of their faces are quadrilaterals (each plane intersects two corresponding pairs of opposite faces of the parallelepiped and does not intersect the remaining two faces). One of the hexahedrons has a circumscribed sphere. Prove that each of these hexahedrons has a circumscribed sphere.

5 [**8**] Let $\binom{n}{k}$ be the number of ways that k objects can be chosen (regardless of order) from a set of n objects. Prove that if positive integers k and l are greater than 1 and less than n, then integers $\binom{n}{k}$ and $\binom{n}{l}$ have a common divisor greater than 1.

6 [**9**] An integer $n > 1$ is given. Two players in turns mark points on a circle. First Player uses red color while Second Player uses blue color. The game is over when each player marks n points. Then each player finds the arc of maximal length with ends of his color, which does not contain any other marked points. A player wins if his arc is longer (if the lengths are equal, or both players have no such arcs, the game ends in a draw). Which player has a winning strategy?

7 [**9**] Initially a number 6 is written on a blackboard. At n-th step an integer k on the blackboard is replaced by $k+\gcd(k,n)$. Prove that at each step the number on the blackboard increases either by 1 or by a prime number.

[1] Courtesy of Andy Liu