International Mathematics

TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Spring 2009.**[1]

1 [**3**] In a convex 2009-gon, all diagonals are drawn. A line intersects the 2009-gon but does not pass through any of its vertices. Prove that the line intersects an even number of diagonals.

2 [**4**] Let a^b denote the number a^b. The order of operations in the expression 7^7^7^7^7^7^7 must be determined by parentheses (5 pairs of parentheses are needed). Is it possible to put parentheses in two distinct ways so that the value of the expression be the same?

3 [**4**] Alex is going to make a set of cubical blocks of the same size and to write a digit on each of their faces so that it would be possible to form every 30-digit integer with these blocks. What is the minimal number of blocks in a set with this property? (The digits 6 and 9 do not turn one into another.)

4 [**4**] We increased some positive integer by 10% and obtained a positive integer. Is it possible that in doing so we decreased the sum of digits exactly by 10%?

5 [**5**] In rhombus $ABCD$, angle A equals 120°. Points M and N are chosen on sides BC and CD so that angle NAM equals 30°. Prove that the circumcenter of triangle NAM lies on a diagonal of of the rhombus.

[1]Courtesy of Andy Liu

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Senior O-Level Paper **Spring 2009**

1 [**3**] Let $a\hat{}b$ denote the number a^b. The order of operations in the expression $7\hat{}7\hat{}7\hat{}7\hat{}7\hat{}7\hat{}7$ must be determined by parentheses (5 pairs of parentheses are needed). Is it possible to put parentheses in two distinct ways so that the value of the expression be the same?

2 [**4**] Several points on the plane are given; no three of them lie on the same line. Some of these points are connected by line segments. Assume that any line that does not pass through any of these points intersects an even number of these segments. Prove that from each point exits an even number of the segments.

3 For each positive integer n, denote by $O(n)$ its greatest odd divisor. Given any positive integers $x_1 = a$ and $x_2 = b$, construct an infinite sequence of positive integers as follows: $x_n = O(x_{n-1} + x_{n-2})$, where $n = 3, 4, \ldots$.

(a) [**2**] Prove that starting from some place, all terms of the sequence are equal to the same integer.

(b) [**2**] Express this integer in terms of a and b.

4 [**4**] Several zeros and ones are written down in a row. Consider all pairs of digits (not necessarily adjacent) such that the left digit is 1 while the right digit is 0. Let M be the number of the pairs in which 1 and 0 are separated by an even number of digits (possibly zero), and let N be the number of the pairs in which 1 and 0 are separated by an odd number of digits. Prove that $M \geq N$.

5 [**4**] Suppose that X is an arbitrary point inside a tetrahedron. Through each vertex of the tetrahedron, draw a straight line that is parallel to the line segment connecting X with the intersection point of the medians of the opposite face. Prove that these four lines meet at the same point.