International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring** 2009

1. Basil and Peter play the following game. Initially, there are two numbers on the blackboard, $\frac{1}{2009}$ and $\frac{1}{2008}$. At each move, Basil chooses an arbitrary positive number x, and Peter selects one of the two numbers on the blackboard and increases it by x. Basil wins if one of the numbers on the blackboard increases to 1. Does Basil have a winning strategy, regardless of what Peter does?

2. (a) Prove that there exists a polygon which can be dissected into two congruent parts by a line segment which cuts one side of the original polygon in half and another side in the ratio 1:2.

   (b) Can such a polygon be convex?

3. The central square of an 101×101 board is the bank. Every other square is marked S or T. A bank robber who enters a square marked S must go straight ahead in the same direction. A bank robber who enters a square marked T must make a right turn or a left turn. Is it possible to mark the squares in such a way that no bank robber can get to the bank?

4. In a sequence of distinct positive integers, each term except the first is either the arithmetic mean or the geometric mean of the term immediately before and the term immediately after. Is it necessarily true that from a certain point on, the means are either all arithmetic means or all geometric means?

5. A castle is surrounded by a circular wall with 9 towers. Some knights stand on guard on these towers. After every hour, each knight moves to a neighbouring tower. A knight always moves in the same direction, whether clockwise or counter-clockwise. At some hour, there are at least two knights on each tower. At another hour, there are exactly 5 towers each of which has exactly one knight on it. Prove that at some other hour, there is a tower with no knights on it.

6. In triangle ABC, $AB = AC$ and $\angle CAB = 120°$. D and E are points on BC, with D closer to B, such that $\angle DAE = 60°$. F and G are points on AB and AC respectively such that $\angle FDB = \angle ADE$ and $\angle GEC = \angle AED$. Prove that the area of triangle ADE is equal to the sum of the areas of triangles FBD and GCE.

7. Let $\binom{n}{k}$ be the number of ways of choosing a subset of k objects from a set of n objects. Prove that if k and ℓ are positive integers less than n, then $\binom{n}{k}$ and $\binom{n}{\ell}$ have a common divisor greater than 1.

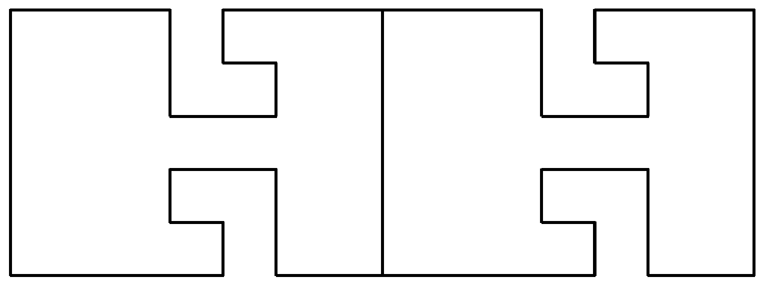
Note: The problems are worth 3, 2+3, 5, 5, 6, 7 and 9 points respectively.

Courtesy of Andy Liu

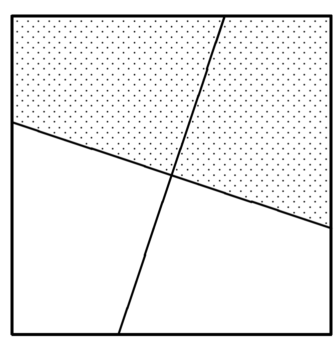
Solution to Junior A-Level Spring 2009

1. Basil starts by choosing the number $\frac{2007}{2008}$. If Peter adds it to $\frac{1}{2008}$, Basil wins immediately. Hence he must add it to $\frac{1}{2009}$, yielding $\frac{4034071}{4034072}$. Basil now chooses $\frac{1}{4034072}$, and Peter can only add it to the number which is not $\frac{4034071}{4034072}$. However, when this is done 4032062 times, the other number will also become $\frac{4034071}{4034072}$. Basil now wins by choosing $\frac{1}{4034072}$ once more.

2. (a) The following diagram shows such a polygon along with the dissecting line.



   (b) **Solution by Daniel Spivak:**
   Divide the sides of a square in counter-clockwise order in the ratio 1:2. If we connect both pairs of points of division on opposite sides, the square is dissected into four congruent parts. If we connect only one pair, we have two congruent convex quadrilaterals. Disregard one of them, and the line connecting the other pair of points of divisioni will dissect the remaining convex quadrilateral into two congruent parts.



3. **Solution by Yung-lin Yang:**
   The robber plans ahead for his escape route. Divide the square into 51 layers of concentric squares. The bank is the sole square of the 0-th layer. The eight surrounding squares constitute the 1-st layer, and so on. With no movement restriction while in the bank, the robber can get to the 1-st layer. Suppose the robber gets to the n-th layer. If the square of entry is marked S, he can go straight into the $(n+1)$-st layer. If the square is marked T, he turns either way and heads for a corner. If he passes on his way a square marked T, he can turn and get to the $(n+1)-st$ layer. If this does not happen by the time he gets to a corner of the n-th layer, he can go to the $(n+1)-st$ layer regardless of whether the corner square is marked S or T. It follows that he can leave the 101×101 square. The bank robber then realizes that the escape route can be traversed in the opposite direction and leads him from outside to the bank!

4. **Solution by Daniel Spivak and Yu Wu, independently:** In the sequence defined by $a_{2k-1} = k^2$ and $a_{2k} = k(k+1)$ for all $k \geq 1$, we have

$$\sqrt{a_{2k-1}a_{2k+1}} = \sqrt{k^2(k+1)^2} = k(k+1) = a_{2k}$$

while

$$\frac{1}{2}(a_{2k} + a_{2k+2}) = \frac{1}{2}(k(k+1) + (k+1)(k+2)) = (k+1)^2 = a_{2k+1}.$$

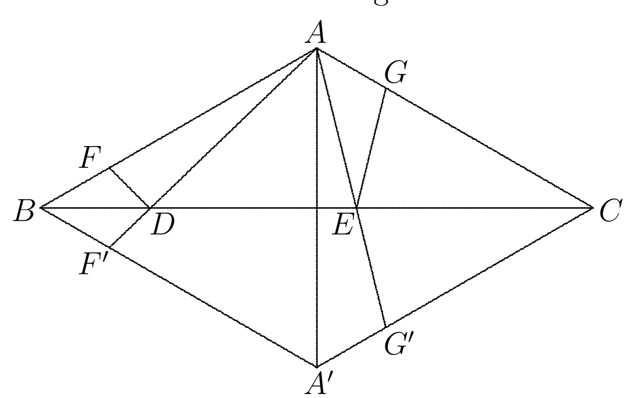
Hence the means are alternately geometric and arithmetic.

5. **Solution by Olga Ivrii and Yung-lin Yang:**
Knights who move together formed a group. At the hour when there are exactly 5 towers each of which has exactly one knight on it, each of these 5 knights is a group by himself. Since there are at most 2 groups on each of the other 4 towers, the total number of groups is at most 13. Suppose the number of groups moving clockwise is no less than the number of groups moving counter-clockwise. Let the former remain in place and let the latter skip over one tower and move to the one beyond. We call them stationary groups and skipping groups respectively. We have exactly the same distribution of knights among the towers. Suppose there is a stationary group on each tower. At the hour when there are exactly 5 towers each of which has exactly one knight on it, each of these 5 knights is a stationary group. Now there can be at most 4 moving groups. So at any hour, one of these 5 towers will have exactly one knight on it, contradicting the condition that at some hour, there are at least two knights on each tower. Hence there is at least one tower with no stationary groups. Each skipping group can visit it once every 9 hours. Since the number of skipping groups is at most 6, this tower will be unguarded at some hour.

6. We use the symbol [ ] to denote area. Reflect the diagram about BC so that A', F' and G' are the respective images of A, F and G. Then D lies on AF' and E lies on AG', and both ABA' and ACA' are equilateral triangles. Now

$$\angle A'AG' = \angle DAE - \angle F'AG' = \angle BAA' - \angle F'AG' = \angle BAF'.$$

It follows that triangles BAF' and $A'AG'$ are congruent.



We have

$$[ADE] + [DEG'A'F'] = [AF'A'] + [AA'G'] = [AF'A'] + [BAF'] = [BAA'] = \frac{1}{2}[BACA'].$$

On the other hand, triangles BDF and CEG are congruent respectively to triangles BDF' and CEG'. Hence

$$[BDF] + [CEG] + [DEG'A'F'] = [BDF'] + [CEG'] + [DEG'A'F'] = [BCA'] = \frac{1}{2}[BACA'].$$

It follows that $[ADE] = [BDF] + [CEG]$.

7. **Solution by Jonathan Schneider:**
Let $0 < k < \ell < n$. Then $\binom{\ell}{k} < \binom{n}{k}$. Suppose we have n players from which we wish to choose a team of size ℓ, and to choose k captains among the team players. The team can be chosen in $\binom{n}{\ell}$ ways and the captains can be chosen in $\binom{\ell}{k}$ ways. On the other hand, if we choose the captains first among all the players, the number of ways is $\binom{n}{k}$. From the remaining $n-k$ players, there are $\binom{n-k}{\ell-k}$ ways of choosing the $\ell - k$ non-captain players. Hence $\binom{n}{\ell}\binom{\ell}{k} = \binom{n}{k}\binom{n-k}{\ell-k}$. Now $\binom{n}{k}$ divides $\binom{n}{\ell}\binom{\ell}{k}$. If it is relatively prime to $\binom{n}{\ell}$, then it must divide $\binom{\ell}{k}$. This is a contradiction since $\binom{\ell}{k} < \binom{n}{k}$.