**International Mathematics**
**TOURNAMENT OF THE TOWNS**

**Senior O-Level Paper** **Fall 2009.**[1]

1. A 7-digit passcode is called good if all digits are different. A safe has a good passcode, and it opens if seven digits are entered and one of the digits matches the corresponding digit of the passcode. Is there a method of opening the safe box with an unknown passcode using less than 7 attempts?

2. $A$, $B$, $C$, $D$, $E$ and $F$ are points in space such that $AB$ is parallel to $DE$, $BC$ is parallel to $EF$, $CD$ is parallel to $FA$, but $AB \neq DE$. Prove that all six points lie in the same plane.

3. Are there positive integers $a$, $b$, $c$ and $d$ such that $a^3 + b^3 + c^3 + d^3 = 100^{100}$?

4. A point is chosen on each side of a regular 2009-gon. Let $S$ be the area of the 2009-gon with vertices at these points. For each of the chosen points, reflect it across the midpoint of its side. Prove that the 2009-gon with vertices at the images of these reflections also has area $S$.

5. A country has two capitals and several towns. Some of them are connected by roads. Some of the roads are toll roads where a fee is charged for driving along them. It is known that any route from the south capital to the north capital contains at least ten toll roads. Prove that all toll roads can be distributed among ten companies so that anybody driving from the south capital to the north capital must pay each of these companies.

**Note:** The problems are worth 4, 4, 4, 4 and 5 points respectively.

[1] Courtesy of Andy Liu.

**Solution to Senior O-Level Fall 2009**

1. In six attempts, try entering 0123456, 0234561, 0345612, 0456123, 0561234 and 0612345. Since the correct passcode uses 7 different digits, it must use at least 3 of the digits 1, 2, 3, 4, 5 and 6. At most one of these 3 can be in the first place. The other 2 must match one of our attempts.

2. Suppose to the contrary the six points do not all lie in the same plane. Now $B$, $C$ and $D$ determine a plane, which we may assume to be horizontal. Suppose that $E$ does not lie in this plane. Since $AB$ is parallel to $DE$ , $A$ does not lie in this plane either. Since $AB \neq DE$, $A$ and $E$ do not lie in the same horizontal plane. Since $BC$ is parallel to $EF$, $F$ lies on the same horizontal plane as $E$. Since $CD$ is parallel to $FA$, $A$ lies on the same horizontal plane as $F$. This is a contradiction. It follows that $E$ also lies on the horizontal plane determined by $B$, $C$ and $D$. Since $BC$ is parallel to $EF$, $F$ also lies in this plane, and since $FA$ is parallel to $CD$, $A$ does also.

3. For $a = 10^{66}$, $b = 2a$, $c = 3a$ and $d = 4a$, $a^3 + b^3 + c^3 + d^3 = (1^3 + 2^3 + 3^3 + 4^3)(100^{33})^3 = 100^{100}$,

4. Let 1 be the side length of the regular 2009-gon $A_1A_2 \ldots A_{2009}$. For indexing purposes, we treat 2010 as 1. For $1 \leq k \leq 2009$, let $B_k$ be the chosen point on $A_kA_{k+1}$ with $A_{2010} = A_1$, $C_k$ be the image of reflection of $B_k$, and $d_k = A_kB_k$. Let $S = d_1+d_2+\cdots+d_{2009}$ and $T = d_1d_2+d_2d_3+\cdots + d_{2009}d_1$. Now $B_1B_2 \ldots B_{2009}$ may be obtained from the regular 2009-gon by removing 2009 triangles, each with an angle equal to the interior $\theta$ angle of the regular 2009-gon, flanked by two sides of lengths $1 - d_k$ and $d_{k+1}$. Hence its area is equal to that of the regular 2009-gon minus $\frac{1}{2} \sin\theta$ times $(1 - d_1)d_2 + (1 - d_2)d_3 + \cdots + (1 - d_{2009})d_1 = S - T$. Similarly, the area of $C_1C_2 \ldots C_{2009}$ is equal to that of the regular 2009-gon minus $\frac{1}{2} \sin\theta$ times $d_1(1-d_2)+d_2(1-d_3)+\cdots+d_{2009}(1-d_1) = S-T$. Hence these two 2009-gons have the same area.

5. List all possible routes from the south capital to the north capital and index them 1, 2, 3, .... Label the first toll road on each route 1. Now the first toll road in route $k$ may also be a later toll road in another route. Label this toll road in the other route 1, modified to $1(k)$ to

keep track of why it is so labelled. All toll roads on a route between two labelled 1 are also labelled 1. This may trigger further labelling and prolong the round, but at some point, this must terminate. Now label the first unlabelled toll road on each route 2, and so on, until all toll roads have been labelled. We continue the modification process to keep track of on which route a certain label first appears. Note that along each route, the labels on the toll roads either remain the same or increase by 1. Assign all toll roads labelled $\ell$ to the $\ell$-th company. We claim that each route has at least one toll road labelled 10. Assume that the highest label of a toll road on a certain route $k_1$ is less than 10. If each label appears exactly once on this route, then it has less than 10 toll roads, which is a contradiction. Hence some label appears more than once. Let the highest label which appears more than once be $h_1$, and consider the last time it appears. It must have been modified to $h_1(k_2)$ for some route $k_2$. We now follow $k_2$ until this toll road, and then switch to $k_1$. This combination must be one of the listed routes, say $k_3$.

Now the highest label of a toll road on this route is also less than 10. Hence some label appears more than once, and such a label must be less than $h_1$. Let the highest label which appears more than once be $h_2$, and consider the last time it appears. It must have been modified to $h_2(k_4)$ for some route $k_4$. We now follow $k_4$ until this toll road, and then switch to $k_3$. Continuing this way, we will find a route in which every label appears exactly once, and the highest label is less than 10. This is a contradiction.