International Mathematics

TOURNAMENT OF THE TOWNS

Solutions to Seniors A-Level Paper **Fall 2009**

1. A pirate who owes money is put in group A, and the others are put in group B. Each pirate in group A puts the full amount of money he owes into a pot, and the pot is shared equally among all 100 pirates. For each pirate in group B, each of the 100 pirates puts 1/100-th of the amount owed to him in a pot, and this pirate takes the pot. We claim that all debts are then settled. Let a be the total amount of money the pirates in group A owe, and let b be the total amount of money owed by the pirates in group B. Clearly, $a = b$. Each pirate in group A pays off his debt, takes back a/100 and then pays out another $b/100$. Hence he has paid off his debt exactly. Each pirate in group B takes in $a/100$, pays out $b/100$ and then takes inwhat is owed him. Hence the debts to him have been settled too. (Wen-Hsien Sun)

2. Let the given rectangle R have length m and width n with $m > n$. Contract the length of R by a factor of n/m, resulting in an $n \times n$ square. For each of the N rectangle in R, the corresponding rectangle in S has the same width but shorter length. Thus we can cut the former into a primary piece congruent to the latter, plus a secondary piece. Using S as a model, the N primary pieces may be assembled into an $n \times n$ square while the N secondary pieces may be assembled into an $(n - m) \times n$ rectangle. (Rosu Cristina, Jonathan Zung)

3. Let the points of tangency to the sphere of AB, AC, DB and DC be K, L, M and N respectively. The line KL intersects the line BC at some point P not between B and C. By the converse of the undirected version of Menelaus Theorem, since $LA = AK$
$$1 = \frac{BP}{PC} \cdot \frac{CL}{LA} \cdot \frac{AK}{KB} = \frac{BP}{PC} \cdot \frac{CL}{KB}.$$
Since $CL = CN$, $KB = MB$ and $ND = DM$,
$$1 = \frac{BP}{PC} \cdot \frac{CN}{MB} = \frac{BP}{PC} \cdot \frac{CN}{ND} \cdot \frac{DM}{MB}.$$

By the undirected version of Menelaus Theorem, P, M and N are collinear. It follows that K, L, M and N are coplanar, so that KN intersects LM. Similarly, the line joining the points of tangency to the sphere of AD and BC also intersects KN and LM. Since the three lines are not coplanar, they must intersect one another at a single point.

4. Define $f(n) = 1\dots1$ (n 1s) and $f(0) = 1$ so that $[0]! = 1$. Define

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$$

for $0 \le k \le n$.

We use induction on n to prove that $\begin{bmatrix} n \\ k \end{bmatrix}$ is always a positive integer for all $n \ge 1$. For $n = 0$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \frac{[0]!}{[0]![0]!} = 1.$$

Suppose the result holds for some $n \ge 0$. Consider the next case.

$$\begin{aligned}
\begin{bmatrix} n+1 \\ k \end{bmatrix} &= \frac{[n+1]!}{[k]![n+1-k]!} \\
&= \frac{[n]!f(n+1)}{[k]![n+1-k]!} \\
&= \frac{[n]!f(n+1-k)\cdot 10^k}{[k]![n-k]!f(n+1-k)} + \frac{[n]!f(k)}{[k-1]![n+1-k]!} \\
&= 10^k \begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix}.
\end{aligned}$$

Since both terms in the last line are positive integers, the induction argument is complete. In particular, for any positive integers m and n

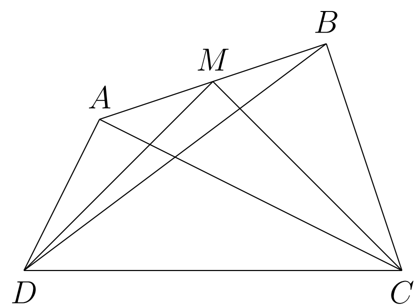
$$\begin{bmatrix} m+n \\ m \end{bmatrix} = \frac{[m+n]!}{[m]![n]!}$$

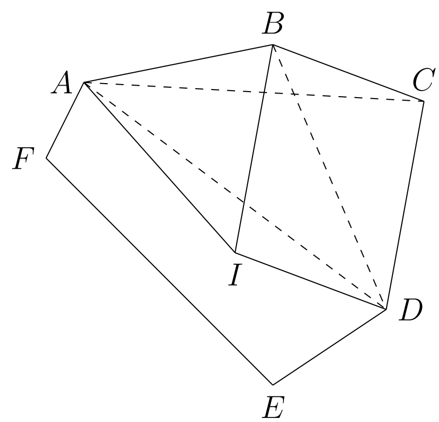
is a positive integer, so that $[m+n]!$ is divisible by $[m]![n]!$. (Jonathan Zung)

5. Denote the area of a polygon P by $[P]$. We first establish

Lemma 1. *Let M be the midpoint of a segment AB which does not intersect another segment CD. Then $[CMD] = ([CAD] + [CBD])/2$.*

Proof. Really, each of three triangles have base CD and height of triangle CMD is average of the sum of the heights of triangles CAD and CBD. □





Returning to the problem, let P, Q and R be the respective midpoints of BC, DE and FA. By the Lemma, we have

$$[PQR] = \frac{1}{2}([BQR]+[CQR]) = \frac{1}{4}([BDR]+[BER]+[CDR]+[CER]) =$$
$$\frac{1}{8}([BAD]+[BFD]+[BAE]+[BFE]+[CAD]+[CFD]+[CAE]+[CFE]).$$

Let I be the point such that triangle ABI is congruent to triangle XYZ. Then $BCDI$ and $EFAI$ are parallelograms. Since $ABCDEF$ is convex, point I is inside the hexagon. Hence $[XYZ] < [ABCDEF]$.

Note that the distance of D from AB is equal to the sum of the distances from C and I to AB; hence, $[BAD] = [BAC] + [BAI] = [BAC] + [XYZ]$.

Similarly, $[BAE] = [BAF] = [XYZ]$. Let J and K be the points such that JCD and FKE are congruent to XYZ. Then we have

$$[ACD] = [BCD] + [XYZ], \qquad [FCD] = [ECD] + [XYZ],$$
$$[BFE] = [AFE] + [XYZ], \qquad [CFE] = [DFE] + [XYZ].$$

It follows that $[PQR] = \frac{1}{2}(2[ABCDEF]+6[XYZ]) > [XYZ]$. (Central Jury)

Remark: The solution above makes a reasonable assumption that triangle XYZ and hexagon $ABCDEF$ are in the same orientation. If they are not then some modifications to the argument are necessary. However, this complication is a mere detraction to an already very nice problem.

6. We construct a graph, with the vertices representing the islands and the edges representing connecting routes. The graph may have one or more connected components. Since the total number of vertices is odd, there must be a connected component with an odd number of vertices.

   Anna chooses from this component the largest set of independent edges, that is, edges no two of which have a common endpoint. She will colour these edges red. Since the number of vertices is odd, there is at least one vertex which is not incident with a red vertex. Anna will start the tour there.

   Suppose Ben has a move. It must take the tour to a vertex incident with a red edge. Otherwise, Anna could have colour one more edge red. Anna simply continues the tour by following that red edge. If Ben continues to go to vertices incident with red edges, Anna will always have a ready response. Suppose somehow Ben manages to get to a vertex not incident with a red edge. Consider the tour so far. Both the starting and the finishing vertices are not incident with red edges. In between, the edges are alternately red and uncoloured. If Anna interchanges the red and uncoloured edges on this tour, she could have obtained a larger independent set of edges.

   This contradiction shows that Ben could never get to a vertex not incident with red edges, so that Anna always wins if she follows the above strategy. (Solution of Central Jury)

7. Answer: As $N = 2^k$, $k = 0, 1, 2, \ldots$

   Solution Let us replace barrels by digits “0”s and “1”s arranged in a circular way.

   (i) Let $N \neq 2^k$, $k = 0, 1, 2, \ldots$. Let us prove that without good luck Ali Baba never opens the cave. We can assume that we are playing against Ali Baba, spinning the circle, and he tells us in advance where he going to change digits in each round, including the first one.

Assume N is odd. Let us place digits "0"s and "1"s to prevent Ali Baba win in the first round. This means that after his first move (which we know) there are both digits "0"s and "1"s.

Let Ali-Baba at some round select k positions. He wins at this round if and only if these k positions coincide either with the set of all "0"s or with the set of all "1"s. But the number of "0"s is not equal to the number of "1"s as their sum is odd and therefore at least one of these numbers differs from k. Then moving such digit to one of selected k positions we prevent Ali Baba from winning at this round.

(ii) The case when N is even but has an odd factor m could be reduced to the previous one. Let us mark on the circle m equidistant positions and forget about all others. Then using the same method as in (i) we can prevent Ali Baba from making digits on these m position equal.

(iii) Algorithm of Ali Baba' win for $N = 2^k$ we construct by induction. The base of induction $k = 1$ is trivial. Let Ali Baba has algorithm A_m for m barrels. Let us construct A_{2m}. All positions for N=2m we split into m pairs of opposite positions . Then we establish one-to-one correspondence between pairs for $N = 2m$ and positions for $N = m$.

First, let us consider the special cases.

(a) Assume that in each pair digits are the same. Then we can apply A_m (with pairs instead of positions). If we unlock the cave in first case ($N = m$) then the cave will be unlocked in second case.

(b) Consider the parity of the sum in each pair. Assume that all parities are the same. Let us apply algorithm A_m; if cave is unlocked then all the parities were odd. However, changing both digits in a pair we do not change the parity of pair, therefore, all the parities remain odd. Let us change exactly one digit in each pair (call this operation D). Then all the parities became even. Applying A_m one more time we unlock the cave. Let us call this algorithm B.

(c) Consider a general case. Let us apply algorithm A_m in the different way: our goal is to make all parities equal. So, we apply A_m but this time changing in each pair only one digit. Call this algorithm C. It guarantees that at some step all parities will coincide. However, we do not know when it will happen. So, we apply the following trick: after each step of applying C we apply B and then D. If after C the parities coincide then B unlocks the cave. Otherwise, BD does not change parities, so parities after C and CBD are the same. As $\underbrace{C\ldots C}_{j\text{ times}}$ makes parities equal and since $\underbrace{CBD\ldots CBD}_{j\text{ times}}$ makes parities equal as well then $\underbrace{CBD\ldots CBD}_{j-1\text{ times}}CB$ unlocks the cave.