International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2009.**[1]

1. Is it possible to cut a square into nine squares and colour one of them white, three of them grey and five of them black, such that squares of the same colour have the same size and squares of different colours will have different sizes?

2. There are forty weights: 1, 2, ..., 40 grams. Ten weights with even masses were put on the left pan of a balance. Ten weights with odd masses were put on the right pan of the balance. The left and the right pans are balanced. Prove that one pan contains two weights whose masses differ by exactly 20 grams.

3. A cardboard circular disk of radius 5 centimetres is placed on the table. While it is possible, Peter puts cardboard squares with side 5 centimetres outside the disk so that:
(1) one vertex of each square lies on the boundary of the disk;
(2) the squares do not overlap;
(3) each square has a common vertex with the preceding one.
Find how many squares Peter can put on the table, and prove that the first and the last of them must also have a common vertex.

4. We only know that the password of a safe consists of 7 different digits. The safe will open if we enter 7 different digits, and one of them matches the corresponding digit of the password. Can we open this safe in less than 7 attempts?

5. A new website registered 2000 people. Each of them invited 1000 other registered people to be their friends. Two people are considered to be friends if and only if they have invited each other. What is the minimum number of pairs of friends on this website?

Note: The problems are worth 3, 4, 4, 5 and 5 points respectively.

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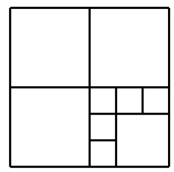
[1]Courtesy of Andy Liu.

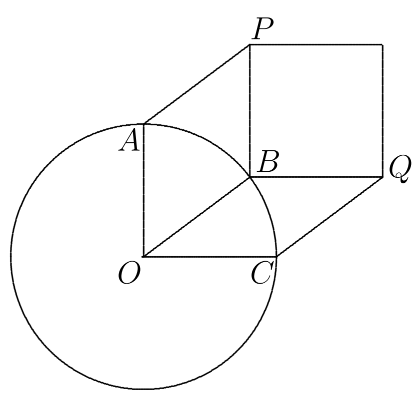
Solution to Junior O-Level Fall 2009

1. The diagram below shows that a 6×6 square can be cut into one 2×2 square, three 3×3 squares and five 1×1 squares.



2. Suppose to the contrary that no two weights in the same pan differ in mass by exactly 20 grams. Then in the right pan, we must have put in exactly one weight from each of the following ten pairs: (1,21), (3,23), ..., (19,39). The total mass in the right pan is $1+3+\cdots+19+20k = 100+20k$, where k is the number of times we chose the heavier weight from a pair. This is a multiple of 4. Similarly, the total mass in the left pan is $2+4+\cdots+20+20h = 110+20h$, where h is the number of times we chose the heavier weight from a pair. This is not a multiple of 4. We have a contradiction as the two pans cannot possibly balance.

3. Let O be the centre of the circle, A, B and C be the points of contact with the circle of three squares in order, and P and Q be the common vertices of these squares. Call OA, OB and OC the root canals of the respective squares. Then $OAPB$ and $OBQC$ are rhombi. Moreover, $\angle PBQ = 90°$. Hence $\angle AOC = 90°$. This means that every two alternate root canals are perpendicular. It follows that there must be 8 root canals, and the last square must have a common vertex with the first.



4. In six attempts, we enter 0123456, 0234561, 0345612, 0456123, 0561234 and 0612345. Since the password uses 7 different digits, it must use at

least 3 of the digits 1, 2, 3, 4, 5 and 6. At most one of these 3 can be in the first place. The other 2 must match one of our attempts.

5. Pretend that the 2000 people are seated at a round table, evenly spaced. Each invites the next 1000 people in clockwise order. Then only two people who are diametrically opposite to each other become friends. This shows that the number of pairs of friends may be as low as 1000. Construct a directed graph with 2000 vertices representing the people. Each vertex is incident to 1000 outgoing arcs representing the invitations. The total number of arcs is 2000×1000. The total number of pairs of vertices is $2000 \times 1999 \div 2 = 1999 \times 1000$. Even if every pair of vertices is connected by an arc, we still have $2000 \times 1000 - 1999 = 1000$ extra arcs. These can only appear as arcs going in the opposite direction to existing arcs. It follows that there must be at least 1000 reciprocal invitations, and therefore at least 1000 pairs of friends.