International Mathematics

TOURNAMENT OF THE TOWNS

Solutions to Junior A-Level Paper **Fall 2009**

1. Pour from each jar exactly one tenth of what it initially contains into each of the other nine jars. At the end of these ten operations, each jar will contain one tenth of what is inside each jar initially. Since the total amount of milk remains unchanged, each jar will contain one tenth of the total amount of milk.

2. Assign spatial coordinates to the unit cubes, each dimension ranging from 1 to 10. If all cubes are in the same colour orientation, there is nothing to prove. Hence we may assume that (i, j, k) and $(i + 1, j, k)$ do not. Since they share a left-right face, let the common colour be red. We may assign blue to the front-back faces of (i, j, k). Then its top-bottom faces are white, the front-back faces of $(i+1, j, k)$ are white and the top-bottom faces of $(i + 1, j, k)$ is blue.

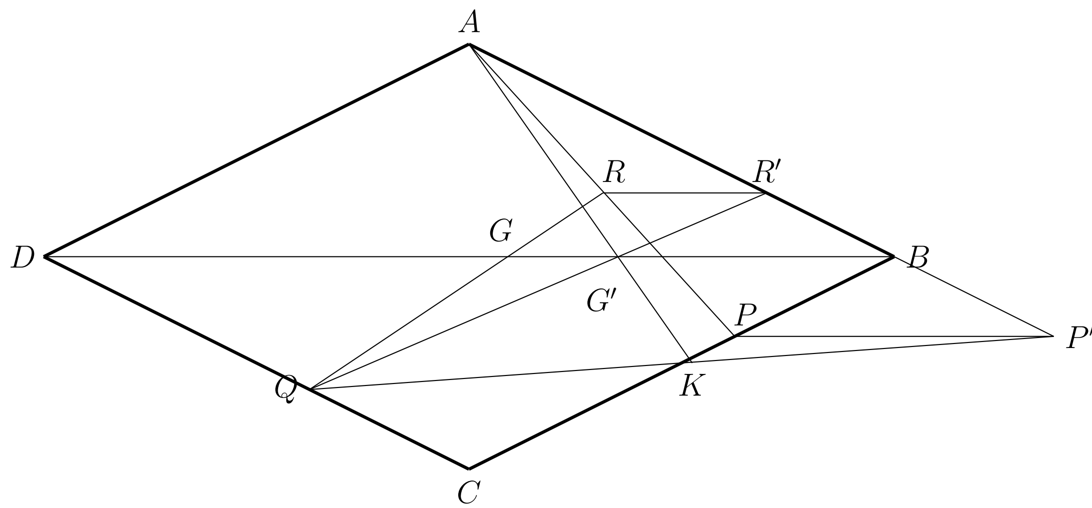
   Now $(i, j + 1, k)$ share a white face with (i, j, k) while $(i + 1, j + 1, k)$ share a blue face with $(i+1, j, k)$. Since $(i, j+1, k)$ and $(i+1, j+1, k)$ share a left-right face, the only available colour is red. It follows that the $1\times2\times10$ block with $(i, 1, k)$ and $(i+1, 1, k)$ at one end and $(i, 10, k)$ and $(i + 1, 10, k)$ at the other end has 1×10 faces left and right which are all red.

   Similarly, if we carry out the expansion vertically, we obtain a $2\times10\times10$ black with 10×10 faces left and right which are all red. Finally, if we carry out the expansion sideways, we will have the left and right faces of the large cube all red.

3. Suppose $a = b$. Then $a+a^2 = a(a+1)$ is a power of 2, so that each of a and $a+1$ is a power of 2. This is only possible if $a = 1$. Suppose $a \neq b$. By symmetry, we may assume that $a > b$, so that $a^2+b > a+b^2$. Since their product is a power of 2, each is a power of 2.

   Let $a^2 + b = 2^r$ and $a + b^2 = 2^s$ with $r > s$. Then $2^s(2^{r-s} - 1) = 2^r - 2^s = a^2 + b - a - b^2 = (a - b)(a + b - 1)$.

   Now $a-b$ and $a+b-1$ have opposite parity. Hence one of them is equal to 2^s and the other to $2^{r-s} - 1$. If $a - b = 2^s = a + b^2$, then -$b = b^2$.



If $a + b - 1 = 2^s = a + b^2$, then $b - 1 = b^2$. Both are contradictions. Hence there is a unique solution $a = b = 1$.

4. Extend AB to P' so that $BP' = BP = CQ$. Then $BP'CQ$ is a parallelogram so that $P'Q$ and BC bisect each other at a point K.

   Let AK intersect BD at G' and let QG' intersect AB at R'. Since K is the midpoint of BC, its distance from BD is half the distance of C from BD, which is equal to the distance of A from BD. It follows that $AG' = 2KG'$.

   Since K is the midpoint of $P'Q$, G' is the centroid of triangle $AP'Q$. Hence $QG' = 2R'G'$ and R' is the midpoint of AP'. Let R be the midpoint of AP and let QR intersect BD at G. Then RR' is parallel to PP', which is in turn parallel to BD. Hence $QG = 2RG$ so that G is the centroid of triangle APQ.

5. (a) Suppose $n+1 = k^2$ for some positive integer k. We take the lightest k objects with total weight $1 + 2 + \cdots + k = k(k+1)/2$ grams. The average weight of the remaining objects is $((k+1) + (k^2 - 1))/2 = k(k+1)/2$ grams also.

   (b) The total weight of the n objects is $1+2+\cdots+n = n(n+1)/2$ grams. Let T grams be the total weight of the k chosen objects. This is also the average weight of the remaining $n - k$ objects. Hence $n(n+1)/2 = T(n-k+1)$.

Now $2T(n-k+1) = n(n+1) > n^2+n-k^2+k = (n+k)(n-k+1)$, so that $2T > n+k$. If we choose the lightest k objects, then T attains its maximum value $((k+1)+n)/2$, so that $2T \leq n+k+1$. It follows that we must have $2T = n+k+1$, and we must take the lightest k objects. Then $(n+k+1)/2 = T = 1+2+\cdots+k = k(k+1)/2$, so that $n+1 = k^2$.

6. Partition the infinite chessboard into $n \times n$ subboards by horizontal and vertical lines n units apart. Within each subboard, assign the coordinates (i,j) to the square at the i-th row and the j-th column, where $1 \leq i,j \leq n$. Whenever an $n \times n$ cardboard is placed on the infinite chessboard, it covers n^2 squares all with different coordinates. The total number of times squares with coordinates $(1,1)$ is covered is 2009. Since 2009 is odd, at least one of the squares with coordinates $(1,1)$ is covered by an odd number of cardboards. The same goes for the other n^2-1 coordinates. Hence the total number of squares which are covered an odd number of times is at least n^2.

7. We construct a graph, with the vertices representing the islands and the edges representing connecting routes. The graph may have one or more connected components. Since the total number of vertices is odd, there must be a connected component with an odd number of vertices.

   Anna chooses from this component the largest set of independent edges, that is, edges no two of which have a common endpoint. She will colour these edges red. Since the number of vertices is odd, there is at least one vertex which is not incident with a red vertex. Anna will start the tour there.

   Suppose Ben has a move. It must take the tour to a vertex incident with a red edge. Otherwise, Anna could have colour one more edge red. Anna simply continues the tour by following that red edge. If Ben continues to go to vertices incident with red edges, Anna will always have a ready response. Suppose somehow Ben manages to get to a vertex not incident with a red edge. Consider the tour so far. Both the starting and the finishing vertices are not incident with red edges. In between, the edges are alternately red and uncoloured. If Anna interchanges the red and uncoloured edges on this tour, she could have obtained a larger independent set of edges.

This contradiction shows that Ben could never get to a vertex not incident with red edges, so that Anna always wins if she follows the above strategy. (Solution of Central Jury)