International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2009** [1]

1 [**4**] Each of 10 identical jars contains some milk, up to 10 percent of its capacity. At any time, we can tell the precise amount of milk in each jar. In a move, we may pour out an exact amount of milk from one jar into each of the other 9 jars, the same amount in each case. Prove that we can have the same amount of milk in each jar after at most 10 moves.

2 [**6**] Mike has 1000 unit cubes. Each has 2 opposite red faces, 2 opposite blue faces and 2 opposite white faces. Mike assembles them into a $10 \times 10 \times 10$ cube. Whenever two unit cubes meet face to face, these two faces have the same colour. Prove that an entire face of the $10 \times 10 \times 10$ cube has the same colour.

3 [**6**] Find all positive integers a and b such that $(a + b^2)(b + a^2) = 2^m$ for some integer m.

4 [**6**] Let $ABCD$ be a rhombus. P is a point on side BC and Q is a point on side CD such that $BP = CQ$. Prove that centroid of triangle APQ lies on the segment BD.

5 We have N objects with weights 1, 2, . . . , N grams. We wish to choose two or more of these objects so that the total weight of the chosen objects is equal to average weight of the remaining objects. Prove that

(a) [**2**] if $N + 1$ is a perfect square, then the task is possible;

(b) [**6**] if the task is possible, then $N + 1$ is a perfect square.

6 [**9**] On an infinite chessboard are placed 2009 $n \times n$ cardboard pieces such that each of them covers exactly n^2 cells of the chessboard. Prove that the number of cells of the chessboard which are covered by odd numbers of cardboard pieces is at least n^2.

7 [**12**] Anna and Ben decided to visit Archipelago with 2009 islands. Some pairs of islands are connected by boats which run both ways. Anna and Ben are playing during the trip:

Anna chooses the first island on which they arrive by plane. Then Ben chooses the next island which they could visit. Thereafter, the two take turns choosing an island which they have not yet visited. When they arrive at an island which is connected only to islands they had already visited, whoever's turn to choose next would be the loser. Prove that Anna could always win, regardless of the way Ben played and regardless of the way the islands were connected.

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[1]Courtesy of Andy Liu