International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2008.**

1. A triangle has an angle of measure θ. It is dissected into several triangles. Is it possible that all angles of the resulting triangles are less than θ, if
   (a) $\theta = 70°$;
   (b) $\theta = 80°$?

2. Alice and Brian are playing a game on the real line. To start the game, Alice places a checker on a number x where $0 < x < 1$. In each move, Brian chooses a positive number d. Alice must move the checker to either $x + d$ or $x - d$. If it lands on 0 or 1, Brian wins. Otherwise the game proceeds to the next move. For which values of x does Brian have a strategy which allows him to win the game in a finite number of moves?

3. A polynomial $x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n$
   has n distinct real roots $x_1, x_2, \ldots, x_n$, where $n > 1$. The polynomial
   $$nx^{n-1} + (n-1)a_1x^{n-2} + (n-2)a_2x^{n-3} + \cdots + 2a_{n-2}x + a_{n-1}$$
   has roots $y_1, y_2, \ldots, y_{n-1}$. Prove that
   $$\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n} > \frac{y_1^2 + y_2^2 + \cdots + y_{n-1}^2}{n-1}.$$

4. Each of Peter and Basil draws a convex quadrilateral with no parallel sides. The angles between a diagonal and the four sides of Peter's quadrilateral are α, α, β and γ in some order. The angles between a diagonal and the four sides of Basil's quadrilateral are also α, α, β and γ in some order. Prove that the acute angle between the diagonals of Peter's quadrilateral is equal to the acute angle between the diagonals of Basil's quadrilateral.

5. The positive integers are arranged in a row in some order, each occuring exactly once. Does there always exist an adjacent block of at least two numbers somewhere in this row such that the sum of the numbers in the block is a prime number?

6. Seated in a circle are 11 wizards. A different positive integer not exceeding 1000 is pasted onto the forehead of each. A wizard can see the numbers of the other 10, but not his own. Simultaneously, each wizard puts up either his left hand or his right hand. Then each declares the number on his forehead at the same time. Is there a strategy on which the wizards can agree beforehand, which allows each of them to make the correct declaration?

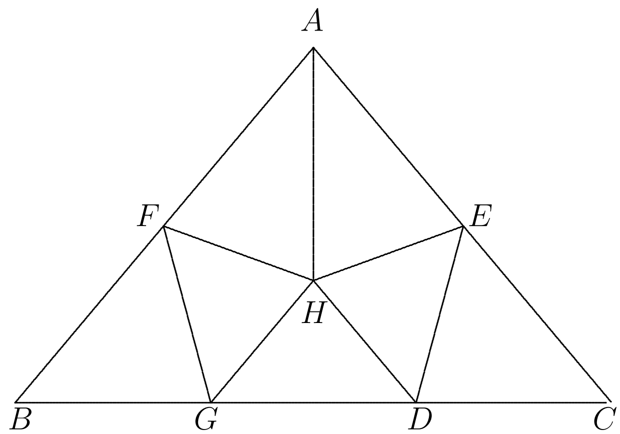
7. Each of three lines cuts chords of equal lengths in two given circles. The points of intersection of these lines form a triangle. Prove that its circumcircle passes through the midpoint of the segment joining the centres of the circles.

Note: The problems are worth 3+3, 6, 6, 7, 8, 8 and 8 points respectively.

Solution to Senior A-Level Spring 2008

1. (a) **Solution by Noble Zhai.**
Suppose the task is possible. In the resulting triangulation, the $70°$ angle must be subdivided into at least two angles. Hence one of these angles is at most $35°$. In the triangle to which it belongs, one of the other two angles is at least $\frac{1}{2}(180° - 35°) = 72.5°$. This is a contradiction.

   (b) In the diagram below, ABC is a triangle with $AB = AC$ and $\angle CAB = 80°$. It is dissected into seven triangles where $AF = AH = AE$, $HF = HG = HD = HE$, HG is parallel to AB and HD is parallel to AC. Then $\angle DHG = 80°$, $\angle HGD = \angle GDH = 50°$, $\angle HAE = \angle HAF = 40°$, $\angle AFH = \angle AHF = \angle FHG = \angle DHE = \angle EHA = \angle AEH = 70°$ and $\angle BGF = \angle CDE = 75°$. The other angles all have measure $55°$. If we move D and G a little closer to each other, we can make all angles to have measure less than $80°$.



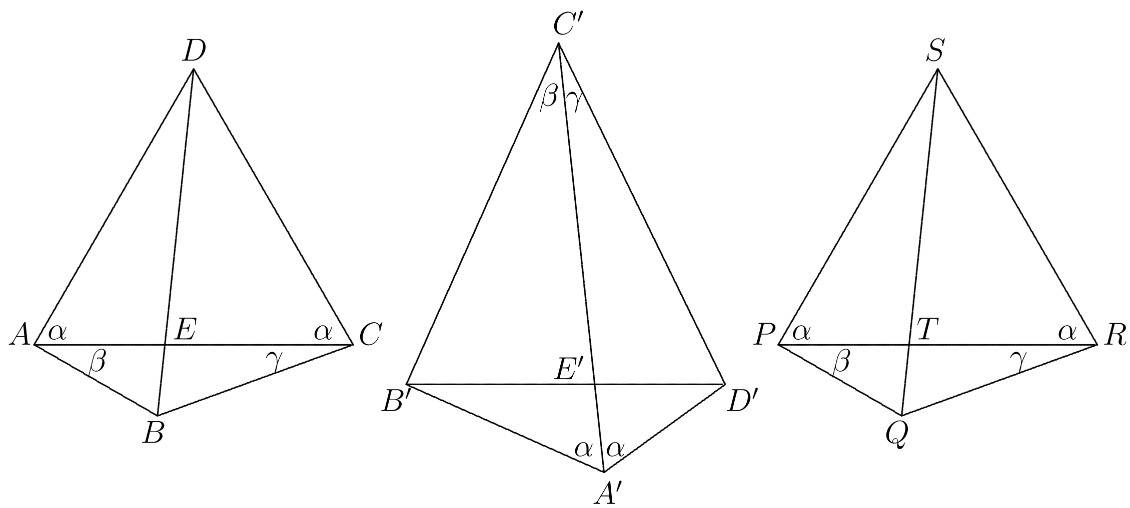
2. Call a number good if it is of the form $\frac{m}{2^n}$ for some odd integer m satisfying $0 < m < 2^n$. Suppose x is a good number. Then Brian chooses $d = \min\{\frac{m}{2^n}, 1 - \frac{m}{2^n}\}$. In order to avoid losing immediately, Alice must move the checker to $\frac{m}{2^{n-1}}$ or $1 - \frac{m}{2^{n-1}}$, which is another good number. Repeating this procedure, the power of 2 in the denominator diminishes by 1 in each move. After n moves, the checker is forced into 0 or 1, and Brian wins. Suppose x is not a good number. Then whatever value d Brian chooses, either $x + d$ or $x - d$ is not good. This is because the sum of two good numbers is good, and half a good number is also good, but x itself is not good. It follows that Alice can never be forced to move the checker to a good number. However, since $\frac{1}{2}$ is good, and it is the only point from which Brian can force Alice to lose on the move, Brian cannot win. In summary, Brian wins if and only if x is a good number.

3. We have $x_1+x_2+\cdots+x_n=-a_1$, $x_1x_2+x_1x_3+\cdots+x_{n-1}x_n=a_2$, $y_1+y_2+\cdots+y_{n-1}=-\frac{a_1(n-1)}{n}$ and $y_1y_2+y_1y_3+\cdots+y_{n-2}y_{n-1}=\frac{a_2(n-2)}{n}$. It follows that $X=x_1^2+x_2^2+\cdots+x_n^2=a_1^2-2a_2$ and $Y=y_1^2+y_2^2+\cdots+y_{n-1}^2=\frac{a_1^2(n-1)^2}{n^2}-\frac{2a_2(n-2)}{n}$. Hence

$$\begin{aligned}\frac{X}{n}-\frac{Y}{n-1} &= a_1^2\left(\frac{1}{n}-\frac{n-1}{n^2}\right)-2a_2\left(\frac{1}{n}-\frac{n-2}{n(n-1)}\right)\\ &= \frac{1}{n^2(n-1)}((n-1)a_1^2-2na_2)\\ &= \frac{1}{n^2(n-1)}((n-1)X-2a_2).\end{aligned}$$

By the Rearrangement Inequality, $(n-1)(x_1^2+x_2^2+\cdots+x_n^2)-2(x_1x_2+x_1x_3+\cdots+x_{n-1}x_n)\geq 0$, with equality if and only if $x_1=x_2=\cdots=x_n$. Since the roots are distinct, we have strict inequality.

4. In the diagram below on the left is Peter's quadrilateral $ABCD$, with $\angle CAD=\angle ACD=\alpha$, $\angle BAC=\beta$ and $\angle ACB=\gamma$, the diagonals intersecting at E. In the diagram below in the middle is Basil's quadrilateral $A'B'C'D'$, with $\angle B'A'C'=\angle D'A'C'=\alpha$, $\angle B'C'A'=\beta$ and $\angle D'C'A'=\gamma$, the diagonals intersecting at E'.



Construct triangle PQT similar to triangle $C'B'E'$. If we extend PT to R, then we have $\angle QTR=\angle D'E'C'$, and we can choose R so that triangle QRT is similar to triangle $D'C'E'$. Similarly, we can extend QT to S so that triangle RST is siimilar to triangle $A'D'E'$. Join SP to complete the quadrilateral $PQRS$, as shown in the diagram above on the right. Now $\angle STP=\angle QTR=\angle D'E'C'=\angle B'E'A'$ and

$$\frac{PT}{ST}=\frac{PT}{QT}\cdot\frac{QT}{RT}\cdot\frac{RT}{ST}=\frac{C'E'}{B'E'}\cdot\frac{D'E'}{C'E'}\cdot\frac{A'E'}{D'E'}=\frac{A'E'}{B'E'}.$$

Hence triangle SPT is similar to triangle $B'A'E'$. It follows that triangle PQR is similar to triangle ABC, and triangle RSP is similar to triangle CDA, so that the quadrilateral $PQRS$ is similar to the quadrilateral $ABCD$. Hence $\angle C'B'E'=\angle PTQ=\angle AEB$.

5. **Solution by Konstantin Matveev.**
Let the first three terms be 1, 4! and 2. So far, each of 1+4!, 4!+2 and 1+4!+2 are composite. We will lengthen the sequence by two terms at a time. The second term to be added is the smallest number not yet in the sequence, and the first term to be added is the factorial of the sum of the preceding terms plus the following term. Thus the fifth term is 3, the fourth term is (1+4!+2+3)!, the seventh term is 4 and the sixth term is (1+4!+2+(1+4!+2+3)!+3+4)!. Since the odd-numbered term added at each stage is the smallest number not yet in the sequence, every positive integer will eventually appear. No positive integer can appear twice as the even-numbered term added at each stage is larger than any number which has been chosen earlier. For any block of consecutive terms other than {1,4!}, the largest term is of the form $n!$ while the sum of the other terms is some positive integer k where $2 \leq k \leq n$. The sum of the terms in the block is $n! + k$, which is composite since it is divisible by k.

6. **Solution by Jonathan Zung.**
Each wizard constructs an 10×10 table, the i-th row being the base-2 representation, with leading 0s, of the number on the forehead of the wizard i places away in clockwise order. Then he computes the sum of the diagonal elements modulo 2. If the sum is 0, he puts up his left hand, and if it is 1, he puts up his right hand. Consider an arbitrary wizard A. Each digit of the base-2 representation of A's number appears on the diagonal of exactly one of the other wizards. Consider the digit which is on the diagonal of wizard B. The other 9 digits on that diagonal are known to A. From B's show of hand, A has the sum of the 10 digits on that diagonal, and he can determine the missing digit. Recovering the other digits in a similar manner, A can reconstruct his own number.

7. **Solution by Dmitri Dziabenko.**
Let the centres of the two circles be L and N, with M the midpoint of LN. Let a line α cut the circles at A_1, A_2, A_3 and A_4 as shown in the diagram below, with $A_1A_2 = A_3A_4$. Let A_5 and A_6 be the respective midpoints of A_1A_2 and A_3A_4. Then LA_5 and NA_6 are both perpendicular to α. Let A_7 be the foot of the perpendicular from M to α. Then A_7 is the midpoint of A_5A_6, and hence of A_2A_3 and of A_1A_4. Now $A_7A_1 \cdot A_7A_2 = A_7A_4 \cdot A_7A_3$. Hence A_7 has equal power with respect to both circles, and lies on their radical axis ℓ. Let β and γ be two other lines which cut the circle in equal segments. Then the feet of the perpendicular from M to both lines also lie on ℓ. It follows that ℓ is the Simson line with respect to M of the triangle formed by α, β and γ. Hence M lies on the circumcircle of this triangle.