International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2008.**

1. A triangle has an angle of measure θ. It is dissected into several triangles. Is it possible that all angles of the resulting triangles are less than θ, if

   (a) $\theta = 70^\circ$;

   (b) $\theta = 80^\circ$?

2. Alice and Brian are playing a game on the real line. To start the game, Alice places a checker on a number x where $0 < x < 1$. In each move, Brian chooses a positive number d. Alice must move the checker to either $x + d$ or $x - d$. If it lands on 0 or 1, Brian wins. Otherwise the game proceeds to the next move. For which values of x does Brian have a strategy which allows him to win the game in a finite number of moves?

3. A polynomial $\quad x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n$

   has n distinct real roots $x_1,\ x_2,\ \ldots,\ x_n$, where $n > 1$. The polynomial

   $$nx^{n-1} + (n-1)a_1x^{n-2} + (n-2)a_2x^{n-3} + \cdots + 2a_{n-2}x + a_{n-1}$$

   has roots $y_1,\ y_2,\ \ldots,\ y_{n-1}$. Prove that

   $$\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n} > \frac{y_1^2 + y_2^2 + \cdots + y_{n-1}^2}{n-1}.$$

4. Each of Peter and Basil draws a convex quadrilateral with no parallel sides. The angles between a diagonal and the four sides of Peter's quadrilateral are $\alpha,\ \alpha,\ \beta$ and γ in some order. The angles between a diagonal and the four sides of Basil's quadrilateral are also $\alpha,\ \alpha,\ \beta$ and γ in some order. Prove that the acute angle between the diagonals of Peter's quadrilateral is equal to the acute angle between the diagonals of Basil's quadrilateral.

5. The positive integers are arranged in a row in some order, each occuring exactly once. Does there always exist an adjacent block of at least two numbers somewhere in this row such that the sum of the numbers in the block is a prime number?

6. Seated in a circle are 11 wizards. A different positive integer not exceeding 1000 is pasted onto the forehead of each. A wizard can see the numbers of the other 10, but not his own. Simultaneously, each wizard puts up either his left hand or his right hand. Then each declares the number on his forehead at the same time. Is there a strategy on which the wizards can agree beforehand, which allows each of them to make the correct declaration?

7. Each of three lines cuts chords of equal lengths in two given circles. The points of intersection of these lines form a triangle. Prove that its circumcircle passes through the midpoint of the segment joining the centres of the circles.

Note: The problems are worth 3+3, 6, 6, 7, 8, 8 and 8 points respectively.