International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring 2008.**

1. An integer N is the product of two consecutive integers.
   (a) Prove that we can add two digits to the right of this number and obtain a perfect square.
   (b) Prove that this can be done in only one way if $N > 12$.
2. A line parallel to the side AC of triangle ABC cuts the side AB at K and the side BC at M. O is the point of intersection of AM and CK. If $AK = AO$ and $KM = MC$, prove that $AM = KB$.
3. Alice and Brian are playing a game on a $1 \times (N + 2)$ board. To start the game, Alice places a checker on any of the N interior squares. In each move, Brian chooses a positive integer n. Alice must move the checker to the n-th square on the left or the right of its current position. If the checker moves off the board, Alice wins. If it lands on either of the end squares, Brian wins. If it lands on another interior square, the game proceeds to the next move. For which values of N does Brian have a strategy which allows him to win the game in a finite number of moves?
4. Given are finitely many points in the plane, no three on a line. They are painted in four colours, with at least one point of each colour. Prove that there exist three triangles, distinct but not necessarily disjoint, such that the three vertices of each triangle have different colours, and none of them contains a coloured point in its interior.
5. Standing in a circle are 99 girls, each with a candy. In each move, each girl gives her candy to either neighbour. If a girl receives two candies in the same move, she eats one of them. What is the minimum number of moves after which only one candy remains?
6. Do there exist positive integers $a,\ b,\ c$ and d such that $\frac{a}{b} + \frac{c}{d} = 1$ and $\frac{a}{d} + \frac{c}{b} = 2008$?
7. A convex quadrilateral $ABCD$ has no parallel sides. The angles between the diagonal AC and the four sides are $55^\circ,\ 55^\circ, 19^\circ$ and 16° in some order. Determine all possible values of the acute angle between AC and BD.

Note: The problems are worth 2+2, 5, 6, 6, 7, 7 and 8 points respectively.

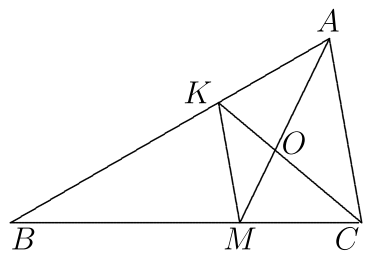
Solution to Junior A-Level Spring 2008

1. Let $N = n(n+1)$. Adding two digits to the right of N produces an integer M where $100n(n+1) \le M \le 100n(n+1)+99$.
   (a) If we add 25, then $M = 100n(n+1)+25 = (10n+5)^2$.
   (b) Notice that $N > 12$ means $n > 3$. It follows that $100n(n+1)-(10n+4)^2 = 20n-16 > 0$ and $100n(n+1)+99-(10n+6)^2 = -20n+63 < 0$. Thus the only square within range is $(10n+5)^2$.

2. Since $AO = AK$, $\angle AKO = \angle AOK$. Since $MK = MC$, $\angle MCK = \angle MKC$. Since KM is parallel to AC, $\angle ACK = \angle MKC$ and $\angle BMK = \angle ACM$. Now

$$\angle ABC = \angle AKO - \angle MCK = \angle AOK - \angle ACK = \angle MAC.$$

Hence triangles MAC and KBM are congruent, so that $AM = BK$.



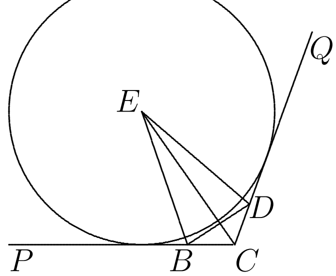
3. Colour the two end squares red. Then there is a block of adjacent blank squares between them. If the block does not have a middle square, the colouring process terminates. If it does, colour the middle square red. This creates twice as many blocks of adjacent blank squares all with the same number of squares in them. Eventually, the colouring process stops either because all squares are red, or because the new blocks do not have middle squares. If all squares are red, then $N = 2^n - 1$ for some positive integer n, and Brian has a sure win. Wherever Alice places the checker, Brian can force it into either square which makes its current square red. Eventually, the checker will be forced into either of the two end squares. If the squares are not all red, then $N \ne 2^n - 1$ for any positive integer n, and Alice cannot lose. She simply places the checker on a blank square, and Brian can never force it onto a red square. Since both end squares are red, Brian cannot win. In summary, Brian wins if and only if $n = 2^n - 1$ for some positive integer n.

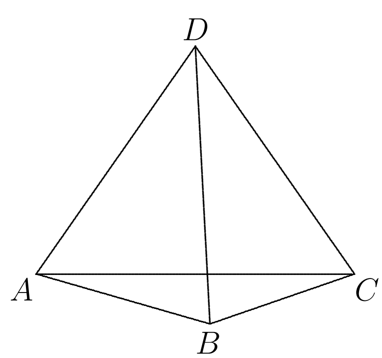
4. Consider all sets of four points of different colours. Since the number of points is finite, we can choose the set whose convex hull has the smallest area. If the convex hull is a quadrilateral, then there are no coloured points in its interior, as otherwise we have a set whose convex hull has smaller area. The four vertices of the quadrilateral determine four triangles each with vertices of different colours, and any three of these four triangles will satisfy the requirement. Suppose the convex hull is a triangle ABC, say with A red, B yellow and C blue. Then only points of the fourth colour, say green, can be inside ABC, and there is at least one such point D. If there are no green points other than D, then triangles ACD, BAD and CBD satisfy the requirement. Suppose BAD contains other green points. Choose among them a point E such that triangle BAE has the smallest area. Then it cannot contain any green points in its interior, and we can replace BAD by BAE. A similar remedy can be applied if either ACD or CBD contains green points in its interior. Hence we will get three triangles which satisfy the requirement.

5. **Solution by Olga Ivrii.**
Let the girls be labelled 1 to 99 in clockwise order. We first show that the task can be accomplished in 98 moves. In each of the first 49 moves, if she still has a candy, the k-th girl gives hers to the $(k-1)$-st girl for $2 \le k \le 50$ and to the $(k+1)$-st girl for $51 \le k \le 99$. The 1-st girl, who will always have a candy, gives hers to the 99-th girl. After 49 moves, only the 1-st and the 99-th girl have candies. These two candies can be passed, in opposite directions, to the 50-th girl in another 49 moves. We now show that the task cannot be accomplished in less than 98 moves. We will not allow the girls to eat the candies, but each must pass all she has to the same neighbour. Our target is to have all candies in the hands of one girl. Consider what happens to a candy in two consecutive moves. It either returns to the girl who has it initially, or is passed to a girl two places away. Suppose the candies all end up with the 50-th girl in at most 98 moves. The number of moves is not enough for the candy initially with the 50-th girl to go once around the circle before returning to her. It follows that the number of moves must be even. However, in order for the candy initially with the 49-th girl to end up in the hands of the 50-th girl in an even number of moves, it must go once around the circle, and that takes 98 moves.

6. Such positive integers exist. We take $b = kd$ for some integer k. Then $a + kc = kd$ and $ka + c = 2008kd$. Solving this system of equations, we have $a = \frac{kd(2008k-1)}{k^2-1}$ and $c = \frac{kd(k-2008)}{k^2-1}$. Taking $d = k^2 - 1$, we have $b = k(k^2-1)$, $a = k(2008k-1)$ and $c = k(k-2008)$ for $k \ge 2009$. When $k = 2009$, we have $a = 8104448639$, $b = 8108484720$, $c = 2009$ and $d = 4036080$.

7. There are three cases to consider, the two equal angles are at the same vertex, they are at opposite vertices but on the same side of the diagonal, and they are at opposite vertices and on opposite sides of the diagonal. The last one can be discarded because there will be two parallel sides. In the first case, as illustrated by the diagram below on the left, let the equal angles be at the vertex C. Draw the excircle of triangle BCD opposite C with centre E. Then A lies on the line CE, and we have $\angle PBE = \angle EBD$ and $\angle QDE = \angle EDB$. Then $\angle BEC = \angle PBE - 55^\circ$ and $\angle DEC = \angle QDE - 55^\circ$. Hence the sum of the angles of triangle BED is $2\angle EBD + 2\angle EDB - 110^\circ = 180^\circ$. Hence $\angle EBD + \angle EDB = 145^\circ$ so that $\angle BED = 35^\circ$. It follows that A and E coincide. By symmetry, we can take $\angle DAC = 16^\circ$. Then $\angle ADB = \angle QDA = 55^\circ + 16^\circ = 71^\circ$. It follows that the acute angle between AC and BD is $71^\circ + 16^\circ = 87^\circ$.





The second case is illustrated by the diagram above on the right, where $\angle DAC = \angle DCA = 55^\circ$, $\angle BAC = 16^\circ$ and $\angle BCA = 19^\circ$. Then $\angle ABC = 180^\circ - 16^\circ - 19^\circ = 145^\circ$. If $DB < DA = DC$, then $\angle ABC = \angle ABD + \angle DBC > \angle BAD + \angle BCD > 145^\circ$. Similarly, if $DB > DA = DC$, then $\angle ABC < 145^\circ$. It follows that $DB = DA = DC$ so that $\angle ABD = 55^\circ + 16^\circ = 71^\circ$. It follows that the acute angle between AC and BD is again $71^\circ + 16^\circ = 87^\circ$. In summary, 87° is the only possible value.