International Mathematics

TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2008.**

1. Arrange the boxes in a line so that the number of cookies in them decreases from left to right. On a sheet of squared paper, draw a "staircase" in which the height of the first column (square side in width) equals the number of cookies in the first box from the left, the height of the second column equals the number of cookies in the second box, and so on. Then the staircase divides into "footsteps": the first footstep (from the left) consists of the highest columns, the second footstep consists of the columns next to the highest, and so on. The last footstep (to the right) consists of the lowest columns. The number of different integers in Alex's records is equal to the number of footsteps of this staircase (the boxes with the maximal number of cookies correspond to the highest footstep, and so on). But this number is equal to the number of different integers in Serge's records. Indeed, choosing a cooky in each box may be described as cutting off the bottom row of cells in our staircase. Therefore, when we fill up the plates with the maximal number of cookies, several rows will be removed so that the lowest footstep will disappear, and thus the number of footsteps will decrease by 1. By filling up the plates with the next to maximal number of cookies, we remove the next footstep, and so on. Hence the number of footsteps equals the number of different integers in Serge's records as required.

2. Answer: $x_1 = 1, x_2 = \ldots = x_n = 0$.

   Solution. Let us square the equality $\sqrt{x_1} + \sqrt{x_2 + \ldots + x_n} = \sqrt{x_2} + \sqrt{x_3 + \ldots + x_n + x_1}$, subtract the sum $x_1 + \ldots + x_n$ from both sides, and square again. We obtain $x_1(x_2 + \ldots + x_n) = x_2(x_3 + \ldots + x_n + x_1)$, hence $(x_1 - x_2)(x_3 + \ldots + x_n) = 0$. Since $x_1 - x_2 = 1$, we have $x_3 + \ldots + x_n = 0$. Since our equations contain square roots of $x_3, \ldots, x_n$, these numbers are nonnegative, and since their sum is 0, each of them is 0.

   Suppose $x_2 \neq 0$, that is, $x_2 - x_3 \neq 0$. Considering the sums which contain $\sqrt{x_2}$ and $\sqrt{x_3}$ and arguing as above, we get $x_1 = 0$. Then

$x_2 = -1$, but since there exists $\sqrt{x_2}$, we obtain a contradiction. Thus $x_2 = 0$, hence $x_1 = 1$, and then all conditions are satisfied.

3. Let B_1, B_2, ..., B_{30} be the midpoints of arcs A_1A_2, A_2A_3, $\ldots A_{30}A_1$ respectively. The area of 60-gon $A_1B_1A_2B_2 \ldots A_{30}B_{30}$ is the sum of the areas of quadrilaterals $OA_1B_1A_2$, $OA_2B_2A_3$, ..., $OA_{30}B_{30}A_1$. But each of these quadrilaterals has perpendicular diagonals, hence the area of each quadrilateral is the half-product of its diagonals. Observe that one of these quadrilaterals can be non-convex (if the center of the circle lies outside the given 30-gon) but its area is calculated in the same way (verify this!). The required sum is then equal to $\frac{1}{2}OB_1 \cdot A_1A_2 + \frac{1}{2}OB_2 \cdot A_2A_3 + \ldots + \frac{1}{2}OB_{30} \cdot A_{30}A_1$. Since $OB_1 = OB_2 = \ldots = OB_{30} = 2$ by the conditions of the problem, this sum is numerically equal to $A_1A_2 + A_2A_3 + \ldots + A_{30}A_1$, as required.

4. Answer: Yes, it can.

   Solution. First take any arithmetic progression of five distinct positive integers, for instance, 1, 2, 3, 4, 5. Their product equals 120 and so is not 2008th power of a positive integer. Multiply each of these numbers by 120^n to obtain 120^n, $2 \cdot 120^n$, $3 \cdot 120^n$, $4 \cdot 120^n$ and $5 \cdot 120^n$. As before, the numbers form an arithmetic progression, and now their product equals 120^{5n+1}. It remains to choose n so that $5n+1$ is divisible by 2008. This is possible, since 5 and 2008 are coprime. We need a y such that $5n+1 = 2008y$. For instance, $y = 2$ and $n = 803$ fit. Then the product is 2008th power of 120^2.

5. We may assume that our rectangles are drawn on an infinite sheet of squared paper. Divide it into squares 2×2 and mark the cells in each square by 1, 2, 3, 4 clockwise starting from the upper left corner. Since both sides of each rectangle are of odd length, its corner cells are marked by the same number. Let us number four different colors by 1, 2, 3, 4 and paint each rectangle with the color whose number marks the corner cells. It is readily seen that the numbers in the corners of any two adjacent rectangles are distinct.