International Mathematics

TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2008**

1 [**3**] Alex distributes some cookies into several boxes and records the number of cookies in each box. If the same number appears more than once, it is recorded only once. Serge takes one cookie from each box and puts them on the first plate. Then he takes one cookie from each box that is still non-empty and puts the cookies on the second plate. He continues until all the boxes are empty. Then Serge records the number of cookies on each plate. Again, if the same number appears more than once, it is recorded only once. Prove that Alex's record contains the same number of numbers as Serge's record.

2 [**3**] Solve the system of equations ($n > 2$)

$$\sqrt{x_1} + \sqrt{x_2 + \cdots + x_n} = \sqrt{x_2} + \sqrt{x_3 + \cdots + x_n + x_1} = \cdots = \sqrt{x_n} + \sqrt{x_1 + \cdots + x_{n-1}};$$
$$x_1 - x_2 = 1.$$

3 [**4**] A 30-gon $A_1A_2\ldots A_{30}$ is inscribed in a circle of radius 2. Prove that one can choose a point B_k on the arc A_kA_{k+1} for $1 \le k \le 29$ and a point B_{30} on the arc $A_{30}A_1$, such that the numerical value of the area of the 60-gon $A_1B_1A_2B_2\ldots A_{30}B_{30}$ is equal to the numerical value of the perimeter of the original 30-gon.

4 [**4**] Five distinct positive integers form an arithmetic progression. Can their product be equal to a^{2008} for some positive integer a?

5 [**4**] On the infinite chessboard several rectangular pieces are placed whose sides run along the grid lines. Each two have no squares in common, and each consists of an odd number of squares. Prove that these pieces can be painted in four colours such that two pieces painted in the same colour do not share any boundary points.